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Modeling Sustainable Development of Cryptocurrencies by a Fractional Pure-Jump Process in DEA Framework

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ABSTRACT

Sustainable cryptocurrency modeling is vital for maximizing both economic and environmental benefits amid significant investor interest. This research develops a comprehensive methodology for cryptocurrency selection by holistically integrating financial aspects, such as returns and risk, with environmental sustainability. To quantify risk and further evaluate cryptocurrency efficiency, we employ an ARMA-GARCH model with fractional normal inverse Gaussian (FNIG) innovations to forecast Value at Risk (VaR) and expected returns. Subsequently, we apply Data Envelopment Analysis (DEA) to identify the most efficient cryptocurrencies, incorporating mining costs and the forecasted VaR as inputs—representing energy cost and risk, respectively—while using the forecasted expected returns as the output. This approach enables a direct comparison of cryptocurrencies based on these critical factors. Our findings demonstrate that accounting for the inherent stochastic behavior of cryptocurrencies leads to more accurate estimations, and the DEA highlights the essential role of energy costs in selecting efficient cryptocurrencies.

1. Introduction

In recent years, authorities around the world have grown increasingly concerned about the energy consumption and climate impact associated with cryptocurrencies. While estimates of these effects may vary, it has been shown that the electric load demand of the Bitcoin network alone could exceed 13 gigawatts, resulting in an annual carbon footprint of over 65 megatons of $CO2$ as of 2021. This level of energy consumption seems to exceed approximately half of the total estimated power demand of

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all global data centres combined, accounting for about half a percent of global electrical energy use [1]. Many investors overlook the challenges of financial modeling, focusing primarily on the potential returns of the digital market. As cryptocurrencies exhibit self-exciting jumps and volatility clustering, they present both opportunity and risks for investors due to the price fluctuations in the market. Moreover, the stochastically continuous property of Levy processes, along with stationary and independent increments, make them well-suited for modeling complex financial phenomena [2]. These models can achieve an appropriate representation of the stylized facts observed in high-frequency data and effectively capture the volatility clustering effect, leptokurtosis, asymmetry, and long-range dependence features. Furthermore, they are not adequately addressed by standard autoregressive conditional heteroskedasticity (ARCH) and generalized ARCH (GARCH) [3] models, particularly when using normal innovations. Although mixture ARMA-GARCH models with generalized hyperbolic distributions [4] can capture the fat-tail property, but fail to fully describe the characteristics of high-frequency returns. In order to deal with this issue, Sun et al. [5] applied a univariate model covering the stylized facts by taking the ARMA-GARCH model with fractional stable distributed residuals. In this regard, the subordinator process was defined [6] with a stochastic integral for the Volterra kernel [7] where it was applied to the innovations on the multivariate ARMA-GARCH model.

Recently, a multivariate ARMA-GARCH model with fractional generalized hyperbolic innovations that is able to capture the stylized facts was introduced by Kim [8]. He defined the fractional generalized hyperbolic process where the non-fractional variant is derived by subordinating time-changed Brownian motion to the generalized inverse Gaussian process. This paper explores a new model for expected return data that is constructed by taking the fractional normal inverse Gaussian (FNIG) innovations on the univariate ARMA-GARCH model. FNIG accounts for asymmetry in the distribution of expected returns, effectively models heavy tails, and enhances forecasting performance by providing a superior fit to historical data [9]. By incorporating fractional dynamics, FNIG helps in capturing long-range dependence, memory effects and makes its structure more resilient to outliers. Given the characteristics of our dataset and the specific application requirements, we have selected the FNIG model. Mba et al. [10] utilized the fractional Ornstein–Uhlenbeck driven by a Normal Inverse Gaussian Levy process model and demonstrated that the NIG distribution offers the best fit for the log returns of Bitcoin data. In addition, we suggest a new investment strategy that not only enables investors to manage their portfolio of various cryptocurrencies in terms of risk and returns, but also considers electric load demand of the assets. This allows investors to construct a portfolio based on efficient assets that align with their preferences. This takes into account the energy consumption required for mining cryptocurrencies, which has become a significant concern due to its environmental impact. Moreover, it guarantees that our investment strategy is both financially optimized and environmentally sustainable. In order to fulfil this aim, we apply data envelopment analysis (DEA) as a decision making tool, which is a non-parametric approach in operations research and economics for calculating production frontiers [11]. DEA has been widely used in a large range of fields from international banking and economic sustainability to operational management in various sectors. To improve the assessment and management of uncertainties, value at risk (VaR) serves as a risk measure in financial studies and daily risk practices. It quantifies the potential loss of a portfolio over a specified time period with a specific confidence level. Since all inputs and outputs of traditional DEA models are assumed to be non-negative, DEA is not applicable to several circumstances, such as analysis of financial statements. To deal with this, for handling negative inputs and/or outputs, we apply range directional model (RDM), which is an extension of DEA [12]. Responding to the growing demand for optimal investments, Mirsadeghpour et al. [13] evaluated portfolio performance using the DEA method. They considered appropriate underlying distributions that influence the model's inputs and outputs, and a comparison of the models' performance shows that accounting for skewness and kurtosis leads to a more interpretable evaluation of efficiency. In this paper, the inputs of the introduced model are the costs of mining and VaR as a risk measure, and the expected return is the only output. By applying DEA to the cryptocurrency market, we can identify the efficient frontier, which represents the optimal combination of cryptocurrencies that maximizes the expected returns and minimizes the risks simultaneous. This allows investors to make their portfolio by selecting the cryptocurrencies that lie on the efficient frontier. Useful simulation and parameter estimation methods are provided, and the goodness-of-fit tests are performed for the estimated parameters.

1.1 Literature Analysis

With the growing prominence of cryptocurrency in various sectors, understanding its research has become increasingly vital. Recent efforts have aimed to illuminate emerging trends on emerging trends and overlooked dimensions within this rapidly evolving field. In this investigation, Alqudah et al. [14] conducted a bibliometric study to examine trends in cryptocurrency research from 2014 to 2021. Their analysis highlighted a limited focus on environmental, social, and governance factors related to the sustainability of cryptocurrency investment. These trends included market efficiency and risk, the rate of adoption, the dominant influence of Bitcoin, the connection between blockchain and sustainability, and investor behavior. While acknowledging the progress made, their main conclusion was that future academic work should prioritize the environmental impacts of consensus mechanisms, comprehensive sustainability assessments, the role of regulations, and the relationship between sustainability and financial performance. This would improve understanding of the long-term viability of cryptocurrency investments. Building on this, Anyssa et al. [15] specifically investigated the complex link between the energy consumption of cryptocurrencies and their environmental consequences. The study addressed challenges in integrating the digital market into supply chains, explored the relationship between media attention on environmental issues and cryptocurrency, and noted promising trends in sustainability. However, the analysis pointed to major challenges in transitioning to lower-energy cryptocurrencies, driven by the profitability and entrenched infrastructure of leading players like Bitcoin. Consequently, a rapid shift driven solely by environmental concerns is unlikely without significant economic drivers and more extensive changes within the broader ecosystem. Recently, Koemtzopoulos et al. [16] analyzed stablecoins as a potential driver of sustainable development, offering environmental benefits, supporting Ethereum's sustainability efforts, and stabilizing the Bitcoin market. Nevertheless, their critical perspective highlights a crucial gap: while stablecoins show promise, ensuring the genuine ecological and ethical sustainability of their underlying assets remains a significant challenge requiring further in-depth investigation. This prevents a definitive judgment on their current contribution to truly sustainable finance, despite ongoing advancements in the regulation of stablecoins. The novelty of this paper lies in its departure from simply restricting investor access to popular yet energy-intensive cryptocurrencies. Instead, it proposes an innovative methodology to empower investors in making informed decisions. This approach centers on a DEA model that strategically incorporates both risk and optimized energy cost as crucial input variables. Simultaneously, the model considers return as the essential output, thereby enabling a comprehensive evaluation of cryptocurrency efficiency. This framework helps investors identify the most efficient cryptocurrencies by considering their risk-return profiles while incorporating the critical aspect of energy consumption. This research presents a comprehensive tool that moves beyond basic exclusionary methods, enhancing cryptocurrency valuation and encouraging investment decisions that balance profitability with resource-conscious practices. This innovative use of DEA in cryptocurrency integrates financial risk and energy efficiency, providing a valuable tool for investors to address the complexities of the market with a focus on sustainability. The rest of this paper is organized as follows.

The structure of the ARMA-GARCH model with FNIG innovations and the corresponding risk measure as the VaR are introduced in section 2. Section 3 is devoted to the data analysis including the estimation method of the ARMA-GARCH parameters, the Hurst index, and the risk measure VaR. In this section, we also evaluate the efficiencies of eight top cryptocurrencies. Finally, the conclusion of the paper is presented in section 4.

2. Methodology

In this section, we employ a methodological framework to analyze financial time series, risk measure, and optimize portfolio selection. The approach integrates advanced econometric models, including the ARMA-GARCH model enhanced with FNIG innovations, to capture volatility dynamics. Additionally, a DEA-based model is utilized to construct optimal portfolios while accounting for efficiency criteria.

2.1 ARMA-GARCH model with FNIG innovations

Financial data often exhibit volatility clustering, a phenomenon effectively captured by GARCH models, enabling more accurate risk assessment and forecasting. The GARCH component accommodates conditional heteroskedasticity, adapting to changes in variance levels, and can incorporate leverage effects. Meanwhile, the ARMA component, with its short memory, complements this by addressing mean dynamics. The combination of ARMA and GARCH models generally yields more reliable forecasts, enabling the simultaneous modeling of both the mean and variance of time series data. This dual modeling capability provides a comprehensive understanding of underlying processes and effectively captures long-range dependence in returns and volatilities. In the context of efficient portfolio management models, the traditional Gaussian assumption can be replaced by the ARMA-GARCH model with alternative innovations and risk measures. In this section, we apply an ARMA-GARCH model with FNIG innovations and the portfolio risk measure VaR to model the dynamics of expected cryptocurrency returns. Specifically, we first present the definition of the FNIG process and then introduce the univariate model of ARMA-GARCH with FNIG innovations.
Let $\{B_H(t), t \geq 0\}$ denotes a fractional Brownian motion with Hurst index (parameter) $0 < H < 1$ and $\{G(t), t \geq 0\}$ an inverse Gaussian process with tail parameter $\alpha > 0$ and skewness parameter $\beta \geq 0$, where $0 \leq |\beta| \leq \alpha$. Subordinating the fractional Brownian motion to the inverse Gaussian process, we have the new process

$$X(t) \stackrel{d}{=} B_H(G(t))$$

as FNIG process with parameters $(H, \alpha, \beta, \delta, \theta)$ where $\delta > 0$ is the scale parameter, $\theta \in \mathbb{R}$ and $\stackrel{d}{=}$ denotes the equality of finite dimensional distributions. It is shown that the FNIG emerges as the continuous limit of a time series with long range dependence and the moments of the process exist. Now, we assume that X be a FNIG$(H, \alpha, \beta, \delta, \theta)$ process which is generated by a Z as a NIG$(\alpha, \beta, \delta, \theta)$ process. The probability density function of Z is given by

$$f_Z(z) = \frac{\alpha\delta K_1(\alpha\sqrt{\delta^2 + (z-\theta)^2})}{\pi\sqrt{\delta^2 + (z-\theta)^2}} e^{\delta\sqrt{\alpha^2-\beta^2}+\beta(z-\theta)},$$

where K_1 is the modified Bessel function of the second kind of order one.

2.2 ARMA-GARCH model construction

ARMA-GARCH models are commonly used to analyze time series data that exhibits volatility clustering and heteroskedasticity [17]. These models combine an ARMA model which captures the au-

tocorrelation structure in the mean and a GARCH model which captures the volatility clustering in the variance of the time series. The innovation term in the ARMA-GARCH model represents the unexpected component of it. Different distributions can be used for the innovation term that covers different features, such as fat tails and skewness. In order to focus on the descriptive performance and risk forecasting performance, we choose the framework of ARMA(1,1)-GARCH(1,1).
Let $P : 0 = t_0 < t_1 < \ldots < t_M = t$ be a partition of interval $[0, t]$ and

$\|P\| = max\{t_j - t_{j-1} | j = 1, 2, \ldots, M\}$.

A one-dimensional discrete-time process $\{Y(t_k), k = 0, 1, 2, \ldots, M\}$ is referred to as a one-dimensional ARMA-GARCH model with FNIG innovations when it is given by the ARMA(1,1)-GARCH(1,1) model as follows

$$\begin{cases} Y(t_{k+1}) = \mu + aY(t_k) + b\sigma(t_k)\varepsilon(t_k) + \sigma(t_{k+1})\varepsilon(t_{k+1}) \\ \sigma^2(t_{k+1}) = w + \xi\sigma^2(t_k)\varepsilon^2(t_k) + \zeta\sigma^2(t_k) \end{cases} \tag{1}$$

where $\varepsilon(t_{k+1}) = X(t_{k+1}) - X(t_k)$ for $k = 0, 1, 2, \ldots$ and $Y(t_0) = 0$ and $\varepsilon(t_0) = 0$. Moreover, a, b, ξ and ζ are the coefficient parameters of the model with standard normal innovations. This combination model covers the volatility effect, fat tails and asymmetric dependence between elements through the NIG process Z, and describes long-range dependence with the FNIG process X. Because $X(t_k)$ can be approximated as

$$X(t_k) \approx \sum_{j=0}^{k-1} K_H(k, j)\big(Z(t_{j+1}) - Z(t_j)\big), \tag{2}$$

where $K_H(k, j)$ is the Volterra kernel which is a function used in the definition of fractional tempered stable motion [8]. This kernel is on the domain $[0, \infty) \times [0, \infty)$ with long-range dependence property where, for $j \leq k$ and $H \in (0, 1)$,

$$K_H(k, j) = c_H\Big((\frac{k}{j})^{H-\frac{1}{2}}(k - j)^{H-\frac{1}{2}} - (H - \frac{1}{2})j^{\frac{1}{2}-H}\int_j^k u^{H-\frac{3}{2}}(u - j)^{H-\frac{1}{2}}du,\Big) \tag{3}$$

in which

$$c_H = \Big(\frac{H(1 - 2H)\Gamma(\frac{1}{2} - H)}{\Gamma(2 - 2H)\Gamma(H + \frac{1}{2})}\Big)^{\frac{1}{2}}.$$

Also, the increment of $X(t_k)$ can be represented as

$$\begin{aligned} X(t_{k+1}) - X(t_k) \approx{}& K_H(k + 1, k)\big(Z(t_{k+1}) - Z(t_k)\big) + \\ & \sum_{j=0}^{k-1}\big(K_H(t_{k+1}, t_j) - K_H(t_k, t_j)\big)(Z(t_{j+1}) - Z(t_j)). \end{aligned} \tag{4}$$

Next, for evaluating the asset performance and selecting a sustainable portfolio, we have to measure the risk based on the ARMA-GARCH model with FNIG innovations. For this, we find the expected return process $\{Y(t_k), k = 0, 1, 2, \ldots, M\}$ for a given discrete time such that $t_= k.\Delta t$ with $\Delta t = t/M$ for $k = 0, 1, 2, \ldots, M$. According to the relation (1),

$$\begin{aligned} Y(t_{k+1}) &= \mu + aY(t_k) + b\sigma(t_k)\varepsilon(t_k) + \sigma(t_{k+1})\frac{X(t_{k+1}) - X(t_k)}{(t_{k+1} - t_k)^H} \\ &= \mu + aY(t_k) + b\sigma(t_k)\varepsilon(t_k) + (\Delta t)^{-H}\sigma(t_{k+1})\big(X(t_{k+1}) - X(t_k)\big). \end{aligned}$$

By the approximation relation of the increments in (4), we have

$$
\begin{aligned}
Y(t_{k+1}) \approx \mu + aY(t_k) + b\sigma(t_k)\varepsilon(t_k) + (\Delta t)^{-H} K_H(k+1,k)\sigma(t_{k+1})\big(Z(t_{k+1}) - Z(t_k)\big) \\
+ (\Delta t)^{-H} \sum_{j=0}^{k-1} \sigma(t_{k+1})\big(K_H(t_{k+1},t_j) - K_H(t_k,t_j)\big)(Z(t_{j+1}) - Z(t_j)). \qquad (5)
\end{aligned}
$$

Let $\big(\mathcal{F}(t_k)\big)_{k=0,1,2,\dots,M}$ denotes the natural filtration generated by Y. Then $\sigma(t_{k+1})$ and the last statement of the above relation are $\mathcal{F}(t_k)$-measurable. Furthermore, since Z has stationary increments, we have

$$
\sigma(t_{k+1})\big(Z(t_{k+1}) - Z(t_k)\big)\big|_{\mathcal{F}(t_k)} \overset{d}{=} \sigma(t_{k+1})Z(\Delta t) \overset{d}{=} \Theta(\Delta t), \qquad (6)
$$

where $\big(\Theta(t)\big)_{t\geq 0}$ is a NIG process.

2.3 Risk measure

To implement the new market model for financial risk management and evaluate the relative efficiency of comparable units represented by different assets, calculating VaR is crucial. VaR is a widely used risk measure that quantifies the financial risk associated with a portfolio or position over a defined time frame and a specified confidence level. This calculation provides valuable insights into the potential losses within a portfolio, enabling informed decision-making in risk assessment and management.

Using ARMA-GARCH models, the VaR for the expected return at time t_{k+1} with information given until time t_k as $\big(\mathcal{F}(t_k)\big)_{k=0,1,2,\dots,M}$ at tail probability level δ is defined by

$$
VaR_\delta\big(Y(t_{k+1}) \mid \mathcal{F}(t_k)\big) = -\inf\{x \in \mathbb{R} | P\big(Y(t_{k+1}) \leq x\big) > \delta\}, \qquad (7)
$$

where $P\big(Y(t_{k+1}) \leq x\big)$ is the conditional probability for $Y(t_{k+1})$ based on the information until time t_k. Thus, the VaR for the ARMA-GARCH model with FNIG innovations can be computed. Using relation (5) and the definition of VaR with respect to filtration $\big(\mathcal{F}(t_k)\big)_{k=0,1,2,\dots,M}$, we obtain

$$
\begin{aligned}
VaR_\delta\big(Y(t_{k+1}) \mid \mathcal{F}(t_k)\big) \approx -\mu + aY(t_k) + b\sigma(t_k)\varepsilon(t_k) \\
- (\Delta t)^{H} K_H(k+1,k) VaR_\delta\Big(\sigma(t_{k+1})\big(Z(t_{k+1}) - Z(t_k)\big)\Big) \\
- (\Delta t)^{-H} \sum_{j=0}^{k-1} \sigma(t_{k+1})\big(K_H(t_{k+1},t_j) - K_H(t_k,t_j)\big)(Z(t_{j+1}) - Z(t_j)).
\end{aligned}
$$

Based on relation (6), the risk measure VaR can be computed as

$$
\begin{aligned}
VaR_\delta\big(Y(t_{k+1}) \mid \mathcal{F}(t_k)\big) \approx -\mu + aY(t_k) + b\sigma(t_k)\varepsilon(t_k) - (\Delta t)^{-H} K_H(k+1,k) VaR_\delta\big(\Theta(\Delta t)\big) \\
- (\Delta t)^{-H} \sum_{j=0}^{k-1} \sigma(t_{k+1})\big(K_H(t_{k+1},t_j) - K_H(t_k,t_j)\big)(Z(t_{j+1}) - Z(t_j)).
\end{aligned}
$$

The VaR represents the maximum potential loss on an investment over a specified period with a given probability. It is calculated using the inverse of the cumulative distribution, which quantifies the probability of a specific loss occurring. In this study, the random variable $\Theta(t)$ follows a NIG process, which is a subclass of infinitely divisible distributions. The integral representation of cumulative distribution, with respect to characteristic function, was investigated by Kim et al. [18]. So, the $VaR_\delta\big(\Theta(\Delta t)\big)$ can be evaluated with respect to characteristic function of $\Theta(t)$, which follows NIG distribution.

2.4 DEA-based portfolio selection models

Data Envelopment Analysis (DEA) is a method for evaluating the relative efficiency of decision-making units. This method is able to consider multiple inputs and outputs simultaneously and provide a clear measure of efficiency, allowing investors to identify which portfolios are performing optimally relative to others. Traditional DEA model typically assumes that all inputs and outputs are non-negative. This can be a limitation in many real-world scenarios where negative values such as losses and costs are present. The range directional model (RDM) has the ability to handle negative data effectively. In this essay, the efficiency scores of each cryptocurrency are calculated using return as the output variable and risk and energy cost as the input variables.
We assume that there are n different cryptocurrecies where the expected return of each asset is defined as $Y^1, Y^2, \ldots, Y^n$, and $Y^j = Y^j(t_k)$, $k = 0, 1, \ldots, M$. For $j = 1, 2, \ldots, n$, $VaR_\delta[Y^j]$ demonstrates the risk measure of j-th asset and also $Cost[Y^j]$ shows the energy consumption of j-th cryptocurrencies. Additionally, Y^o for $o \in \{1, 2, \ldots, n\}$ is considered as an asset under evaluation. Regarding the negative return values, a directional vector as

$$g = (R_{E[Y^o]}, R_{VaR_\delta[Y^j]}, R_{Cost[Y^j]})^T \tag{8}$$

indicates the range of possible improvement. The components of the vector g are

$$\begin{aligned} R_{E[Y^o]} &= \{\max_{j=1,2,\ldots,n} E[Y^j]\} - E[Y^o] \\ R_{VaR_\delta[Y^j]} &= VaR_\delta[Y^o] - \{\min_{j=1,2,\ldots,n} VaR_\delta[Y^j]\} \\ R_{Cost[Y^j]} &= Cost[Y^o] - \{\min_{j=1,2,\ldots,n} Cost[Y^j]\}, \end{aligned}$$

where $E[Y^o]$ is the expected return, $VaR_\delta[Y^o]$ is the risk measure and $Cost[Y^o]$ is the energy consumption of Y^o. This vector represents the direction in which the observed asset or portfolio is projected onto the efficient frontier. For asset under evaluation Y^o and the direction vector g, the following linear model is solved

$$\begin{aligned} \max \quad & \alpha \\ \text{s.t.} \quad & E[Y(\lambda)] \geq E[Y^o] + \alpha R_{E[Y^o]} \\ & VaR_\delta[Y(\lambda)] \leq VaR_\delta[Y^o] - \alpha R_{VaR_\delta[Y^o]} \\ & Cost[Y(\lambda)] \leq Cost[Y^o] - \alpha R_{Cost[Y^o]} \\ & e^T\lambda = 1 \\ & \alpha \geq 0, \lambda \geq 0, \end{aligned} \tag{9}$$

where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)^T$ represents the policy of investing in different proportions of cryptocurrencies and e^T is an unit vector. The α^* as the optimal value of α shows to the inefficiency score of the asset under evaluation. Moreover, in conditions of the above problem, $Y(\lambda) = \Sigma_{j=1}^n \lambda_j Y^j$ and so $E[Y(\lambda)] = \Sigma_{j=1}^n \lambda_j E[Y^j]$, $VaR_\delta[Y(\lambda)] = \Sigma_{j=1}^n \lambda_j VaR_\delta[Y^j]$ and $Cost[Y(\lambda)] = \Sigma_{j=1}^n \lambda_j Cost[Y^j]$. The problem (9) is inspired by RDM model with three constraints.

3. Data analysis

In this section, we present the characteristics of 7 types of cryptocurrency data sets for analyzing the expected returns, including Bitcoin (BTC), Ethereum (ETH), Bitcoin Cash (BCH), Lietcoin (LTC), Cardano (ADA), Dogcoin (DOG), and Ripple (XPR). In the context of fitting an ARMA-GARCH model with fractional normal inverse Gaussian (FNIG) innovations to our data sets, we first determine the model order and then proceed to estimate its parameters. Numerous studies have demonstrated that the GARCH(1,1) model consistently delivers superior forecasting results and aligns with daily stock return series [19]. Furthermore, several studies have effectively utilized the ARAM(1,1)-GARCH(1,1) model [6, 8]. Based on these findings, we choose an order of (1,1) for our model and apply the maximum likelihood estimation (MLE) method to estimate its parameters. Using some statistical methods and tests, we forecast the Value at risk (VaR) as a risk measure and also the expected return by ARMA-GARCH model.

3.1 Parameter estimation

Now, we estimate the parameters of the expected returns for each 7 currencies by the introduced model over the period of 16 May to 4 July, 2024. For this, we employ the following method presented in [6]. In this setting, we use high-frequency data with 1-hour time steps. In this method, first the parameters of the ARMA(1,1)-GARCH(1,1) model with standard normal innovations are estimated by MLE method under the assumption that

$$\sigma^2(t_0) = \frac{w}{1-\xi-\zeta}, \tag{10}$$

where w is the constant and ξ and ζ are the coefficient parameters of the squared volatility of the model. Estimated parameters of ARMA-GARCH models are reported in Table 1.

Table 1
ARMA(1,1)-GARCH(1,1) model with standard normal innovations

ARMA(1,1)	BTC	ETH	ADA	XPR	DOG	BCH	LTC
μ	2.70E-05	-3.00E-04	-1.45E-05	-1.78E-05	2.07E-05	-2.70E-04	2.00E-04
a	0.29	-0.62	-0.63	0.13	0.66	0.0183	-0.35
b	-0.32	0.56	0.65	0.2	-0.74	0.047	0.27
GARCH(1,1)							
w	1.14E-06	9.70E-07	1.38E-06	9.90E-07	3.50E-05	2.55E-06	1.20E-06
ζ	0.66	0.86	0.79	0.75	0.66	0.9	0.78
ξ	0.33	0.13	0.2	0.24	0.09	0.05	0.21

As demonstrated, DOG, a type of cryptocurrency, exhibits distinctive behavior. This asset has the strongest positive autocorrelation in returns with the parameter a in ARMA part of the model 0.66, meaning past positive returns tend to be followed by future positive returns, and vice versa. Additionally, DOG has the highest baseline volatility level with a constant variance term w of the GARCH part as 3.50×10^{-5}, indicating a higher inherent level of price fluctuations. ETH and ADA exhibit a strong mean-reversion effect in returns, with values of -0.62 for ETH and -0.63 for ADA. Both cryptocurrencies also display high volatility persistence, as indicated by the ARCH coefficient (ξ) of 0.13 for ETH and 0.2 for ADA, alongside the GARCH coefficient (ζ) of 0.86 for ETH and 0.79 for ADA. This indicates that periods of high volatility are likely to be followed by more high volatility. In contrast, BCH exhibits the highest volatility persistence, with a ζ value of 0.9, and demonstrates the least reaction to previous volatility shocks, as indicated by its ξ value of 0.05. This suggests that sudden and unexpected price changes in the past have a reduced influence on its current volatility. BTC, XRP, and LTC exhibit relatively persistent volatility, with ζ values of 0.66, 0.75, and 0.78, respectively. They also respond to past return shocks, as indicated by ξ values of 0.33, 0.24, and 0.21, respectively. The key distinction is that BTC exhibits positive autocorrelation in returns, with an a value of 0.29, whereas LTC demonstrates a mean-reversion effect in returns, with an a value of -0.35. The mean coefficients μ are nearly zero for most assets; however, ETH has a value of -3.00×10^{-4}, and LTC has a value of 2.00×10^{-4}, indicating slight tendencies in their average returns. These differences in the ARMA and GARCH coefficients

highlight the unique dynamic characteristics of each asset throughout the analyzed period.
Next, we extract the residuals $\varepsilon(t_k)$ for $k = 1, 2, \ldots, M$ using the estimated parameters. Then, the Hurst index H is estimated by the rescaled range analysis (R/S) method presented in [20]. The R/S is a statistical measure of variability or fractal rescaled ranges where the adjusted range is divided by the standard deviation. The Hurst index estimation is reported in Table 2.

Table 2
Estimated Hurst index for seven cryptocurrencies

Cryptocurrency	Estimated Hurst Index
BTC	0.634
ETH	0.587
ADA	0.574
XPR	0.566
DOG	0.5935
BCH	0.6
LTC	0.567

By the assumption that $X = \{X_{(t_k)}\}_{k=1,2,\ldots,M}$ where $X(t_k) = \sum_{j=1}^{k} \varepsilon(t_j)$, we extract $Z = \{Z(t_k)\}_{k=1,2,\ldots,M}$ as

$$\begin{aligned} Z(t_1) &= \frac{X(t_1)}{K_H(t_1, t_0)}, \\ Z(t_k) &= Z(t_{k-1}) + \frac{X(t_k) - X(t_{k-1})}{K_H(t_k, t_{k-1})} \\ &- \sum_{j=0}^{k-2} \frac{K_H(t_k, t_j) - K_H(t_{k-1}, t_j)}{K_H(t_k, t_{k-1})} \big(Z(t_{j+1}) - Z(t_j)\big). \end{aligned} \tag{11}$$

Given that the estimated Hurst exponent H for each cryptocurrency surpasses 0.5, it can be inferred that they exhibit long-range dependence. We estimate the parameters of the NIG process $\{Z_{(t_k)}\}_{k=1,2,\ldots,M}$ extracted in the last procedure. The estimated parameters are reported in the Table 3.

Table 3
Estimated parameters of FNIG process for seven cryptocurrencies

	BTC	ETH	ADA	XPR	DOG	BCH	LTC
α	6.9	0.1245	5.79	24.014	7.43	5.923	10.77
β	-1.27	-0.209	-0.2958	-1.72	0.0717	-0.3821	0.4721
θ	0.0026	0.0034	0.0023	0.0012	0.0013	0.0021	-0.0024
δ	0.0264	0.0251	0.0351	0.0252	0.0698	0.0315	0.0575

Analysis of the NIG distribution fitting results for Z reveals that digital assets display varying risk and return characteristics, highlighting their distinct behavioral patterns. XRP, with the highest α value 24.014, has the thinnest tails and a lower probability of extreme outlier events, while ETH, with the lowest α value 0.1245, shows the heaviest tails and a higher risk of unexpected volatility. Regarding skewness, LTC and DOG exhibit positive skewness, potentially indicating a higher likelihood of larger gains, though DOG's skewness is closer to zero. In contrast, BTC, ETH, ADA, XRP, and BCH show negative skewness, pointing to an increased probability of larger losses, with XRP and BTC having negative skewness. The location parameter θ is close to zero for most assets, but ETH shows a slight tendency towards higher values, and LTC towards lower values. Additionally, in terms of scale δ, representing volatility, DOG has the highest value 0.0698, indicating the greatest volatility, while ETH and XRP show the lowest volatility in this set. This analysis highlights that each digital asset possesses a unique risk-return profile, emphasizing the importance of factoring these differences into investment strategies. Finally, we employ the RDM approach, presented in Section 3, as a variation of the traditional DEA model that relaxes the assumption of non-negativity of inputs and outputs to measure the efficiency of each cryptocurrency in terms of risk, energy cost, and expected returns as outputs. Figure 1 presents a graphical representation of the processes Z and X for XPR.

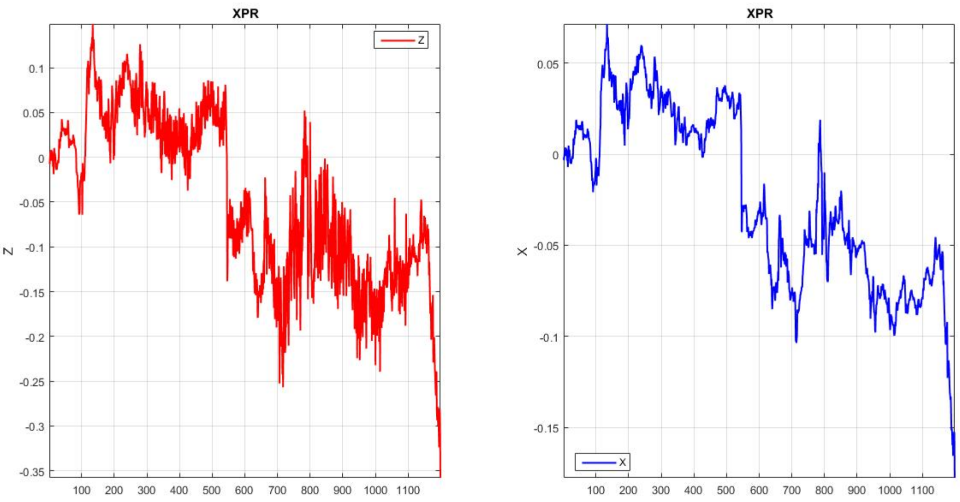

Fig. 1. The processes Z and X for XPR

To compute the goodness-of-fit of the innovation processes, we apply the Kolmogorov–Smirnov (KS) test that is performed based on the following statistic.

$$KS = \sup_{z} |\hat{F}(z) - F(z)|, \tag{12}$$

where $\hat{F}(z)$ and $F(z)$ denote the empirical and theoretical cumulative distribution functions (CDF), respectively. This statistic is calculated for the FNIG distribution with estimated parameters $(\alpha, \beta, \theta, \delta)$ and the empirical distribution of the increments $Z(t_k) - Z(t_{k-1})$ for $k = 1, 2, \ldots, M$.

Table 4
Kolmogorov-Smirnov test for different cryptocurrencies

Cryptocurrency	KS Statistic	p-value
BTC	0.0261	0.3805
ETH	0.0371	0.0948
ADA	0.032	0.1682
XPR	0.0324	0.1571
DOG	0.0374	0.0685
BCH	0.0371	0.0721
LTC	0.0456	0.0133

The p-value associated with the KS statistic quantifies the evidence against the null hypothesis that the data originate from the specified theoretical distribution. The KS statistics and p-values are listed in Table 4. Consequently, smaller p-values provide stronger support for rejecting the null hypothesis.

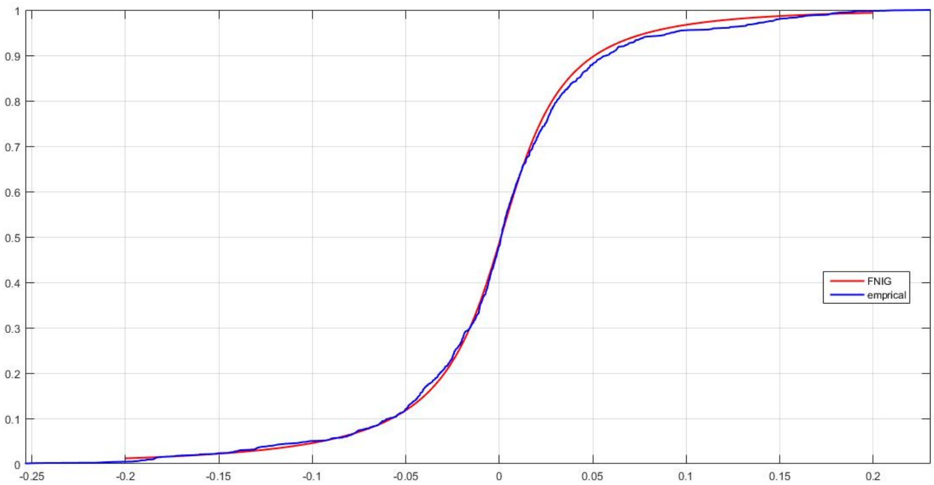

Fig. 2. The CDF of BTC

Figure 2 presents a comparison of the CDF derived from the FNIG model and the empirical CDF.

Table 5
The VaR and expected return of cryptocurrencies

Cryptocurrency	VaR	$E[Y(t_{k+1})\|\mathcal{F}_{t_k}]$
BTC	-0.0078	1.20E-03
ETH	-0.0013	3.79E-04
ADA	-0.0211	0.0153
XPR	-0.0068	2.14E-04
DOG	-0.0131	0.0066
BCH	-0.0197	-2.74E-04
LTC	-0.0260	9.70E-03

3.2 *Forecasting VaR and expected return*

Forecasting both VaR and expected return are crucial for risk management and investment decision making. These forecasts provide investors and financial analysts with essential insights into the potential risks associated with their portfolios and enable them to implement effective strategies to optimize their investment allocations and manage downside risks.

At this point, we forecast a one-hour ahead VaR and expected return using the ARMA-GARCH model with FNIG innovation. Following the introduced estimation method, presented in subsection 4.1, to measure the amount of risk for each currency, the VaR with one percent confidence level is used. The VaR forecast for each currency along with expected return, is presented in Table 5.
To compare the risk measures derived from NIG and FNIG models, we illustrate in Figure 3 that the FNIG model estimates higher VaR forecasts than the NIG model across different confidence levels. This is attributed to the slower decay rate of the FNIG distribution's tail compared to the NIG distribution. The findings derived from Tables 1 to 4 demonstrate alignment with previously published studies, reinforcing the reliability and validity of the model. Specifically, Table 1 underscores the significant and persistent volatility observed within the financial data, reported in prior research. Additionally,

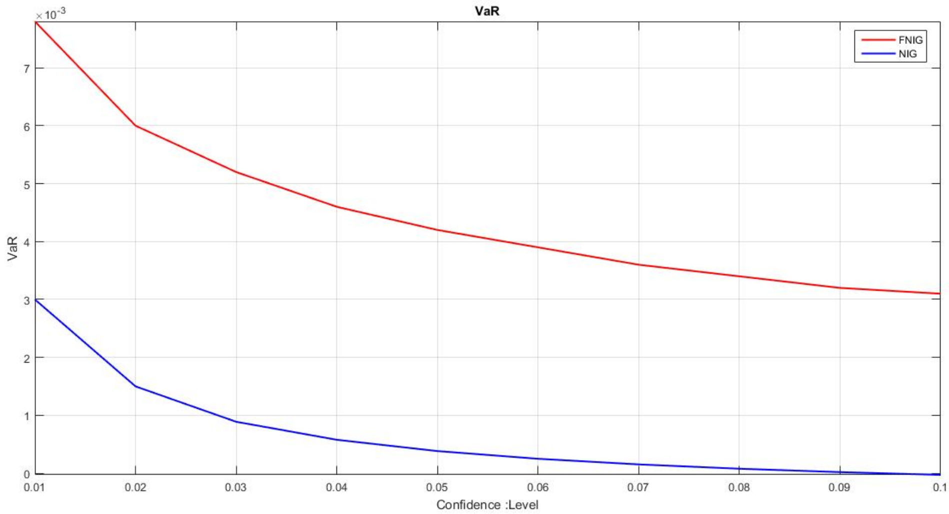

Fig. 3. Forecasting one-hour ahead VaR for BTC using the ARMA-GARCH model with NIG and FNIG innovation

the residuals generated by the ARMA-GARCH model exhibit deviations from normal distribution, as demonstrated in Tables 3 and 4. Furthermore, these residuals display long-range dependency characteristics, as outlined in Table 2, consistent with findings reported in prior studies, including [6, 8]. This study proposes a novel approach using the RDM concept to identify the optimal cryptocurrency. The model leverages energy consumption per transaction (kWh) and VaR forecasts as inputs, with the expected return as the only output. Cryptocurrency efficiency results are detailed in Table 7.

Table 6
Energy consumption per transaction

Cryptocurrency	kWh per transaction
BTC	1173
ETH	87.29
ADA	0.574
XPR	0.0079
DOG	0.12
BCH	18.95
LTC	18.52

The energy consumption per transaction for seven cryptocurrencies is reported in Table 6 from Moneysupermarket, https://www.moneysupermarket.com/gas-and electricity/ features/crypto-energy-consumption. It details the amount of energy, measured in kilowatts, required to process a single transaction for each cryptocurrency. The values indicate variations in energy efficiency across different cryptocurrencies, reflecting their underlying consensus mechanisms and transaction processes. Table 7 presents a comparative analysis of cryptocurrency efficiencies under two distinct scenarios: considering risk alone and incorporating both risk and energy consumption as inputs into the RDM model. When evaluating risk individually, ETH and ADA emerge as the only efficient cryptocurrencies, while BCH is deemed the least efficient. However, the inclusion of energy consumption as an additional factor significantly reshapes the efficiency ranking, as illustrated in Figure 4. XRP and DOG, initially rated below one, achieve a perfect score of one in the second scenario due to their lower energy

Table 7
Comparison of Cryptocurrency Efficiency

Cryptocurrency	Efficiency (risk, return)	Efficiency (risk, energy, return)
BTC	0.62	0.59
ETH	1	0.94
ADA	1	1
XPR	0.53	1
DOG	0.72	1
BCH	0.43	0.56
LTC	0.57	0.62

consumption. BCH and LTC also demonstrate improved efficiency scores under the second criteria, albeit not as dramatically as XRP and DOG. Conversely, BTC's efficiency score decreases, maintaining a consistently lower rating regardless of the evaluation criteria. Finally, ADA consistently maintains its efficient status in both scenarios.

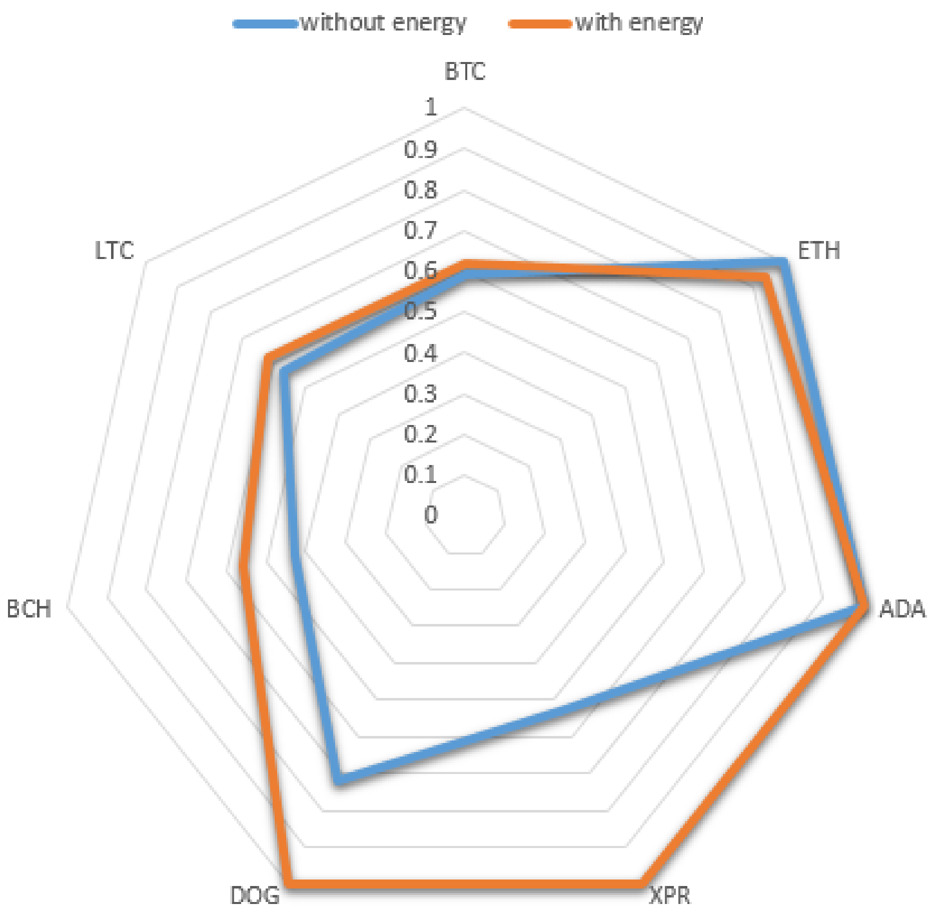

Fig. 4. Comparing efficiency with and without energy consumption

4. Conclusion

The rapidly growing cryptocurrency industry is coming under increasing investigation due to its significant energy consumption and related carbon footprint, creating a challenge for global sustainability efforts. This study introduces a novel framework to empower investors in selecting cryptocurrencies based on their efficiency in managing risk and energy costs simultaneously. To assess cryptocurrency risk, the ARMA-GARCH model driven by FNIG innovation is employed, capturing complexities such as skewness and volatility clustering. Efficiency is then evaluated using RDM, incorporating risk and energy consumption as inputs and expected returns as the output. Results indicate that BTC, despite its market dominance, is inefficient based on risk alone, whereas ETH and ADA are deemed efficient. When factoring in energy consumption, XRP and DOG gain efficiency, highlighting their ability to balance risk, energy use, and returns. Notably, ADA remains efficient across all frameworks, demonstrating favorable risk-return traits with low energy consumption.

The conventional ARMA-GARCH framework exhibits a notable limitation in its capacity to effectively capture the leverage effect, a critical and asymmetric phenomenon frequently observed in financial time series. This constraint highlights the need for alternative approaches that can more accurately account for such dynamics. To overcome this inherent limitation and achieve a more nuanced understanding of how negative shocks disproportionately impact volatility, future research could explore the adoption of advanced models, such as the Glosten-Jagannathan-Runkle GARCH (GJR-GARCH) specification. This approach offers a more robust framework for capturing asymmetries in volatility behavior. Moreover, to enhance the current model and build upon its foundation, extending it to a multi-dimensional framework is strongly recommended. Such an approach would provide a more comprehensive analysis, allowing for the simultaneous modeling that are not addressed in the existing framework. This expansion would facilitate the development of a cryptocurrency portfolio optimized not only for risk and return but also for asset weights, calculated in alignment with their production costs. This objective could be realized through the application of a multi-objective optimization problem.

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