Population Change with Endogenous Birth and Mortality Rates, Wealth Accumulation, and Renewable Resource Change

Wei-Bin Zhang

ABSTRACT

This paper is concerned with dynamic interactions between population change, wealth accumulation, and resource dynamics. Our model synthesizes economic mechanisms of some well-known models within a compact framework. Wealth accumulation is built on the Solow growth model. Dynamics of birth and mortality rates are influenced by the Haavelmo population model and the Barro-Becker fertility choice model. Resource change is modelled on the basis of some growth models with renewable resources. The available time is distributed between work time, leisure time and time of children fostering. We synthesize these dynamic forces in a compact framework by applying an alternative utility function proposed by Zhang. The two-sector model describes a dynamic interdependence between population change, wealth accumulation, and resource dynamics with endogenous time distribution in a perfectly competitive market. We simulate the model to demonstrate existence of equilibrium points and motion of the dynamic system. We also examine effects of changes on the motion of the economic system in the propensity to have children, the propensity to consume goods, the propensity to consume resources, the resource capacity, human capital, and time required for children fostering.

Keywords: propensity to have children, mortality rate, birth rate, renewable resource, wealth accumulation

JEC classification Codes: O41, Q20, J13

Author

Wei-Bin Zhang. Ritsumeikan Asia Pacific University, Beppu-Shi, Japan. Email: wbz1@apu.ac.jp.

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1. Introduction

The purpose of this study is to build a general economic growth model with endogenous population, wealth and renewable resources in a perfectively competitive economy. The model also endogenously determines the time distribution between the work time, leisure time, and time for children fostering. The model is based on different models in the literature economic growth with resources or population. A unique contribution of this paper is to model population growth in a framework of growth with endogenous physical capital accumulation and resource change. The physical capital accumulation is built on the Solow growth model. The neoclassical growth theory based on the Solow growth model is mainly concerned with endogenous physical capital (e.g., Solow, 1956; Burmeister and Dobell, 1970; Azariadis, 1993; Barro and Sala-i-Martin, 1995; and Zhang, 2005). Although there are some extensions of the Solow model by including endogenous population as reviewed late on, this study deviates from the previous studies in modelling birth rate choice and mortality rate. Moreover, as we use the utility function proposed by Zhang (1993). Zhang's approach enables to analyze behavior with decision on leisure time, number of children, consumption levels of goods and resource, and saving. The traditional Solow model lacks economic mechanisms for analyzing complexity of household choice between number of children, time distribution, and (monetary) resource distribution. Except conceptual issues related to the discounting rate of utility in the Ramsey model, our approach simplifies behavior of the representative household far more than the Ramsey approach.

Different economies are experiencing different population phenomena. Some economies are aging rapidly, while some others are faced with Malthusian traps. If we neglect immigration, population change consists of dynamics of birth rate and mortality rate. In modern economies the so-called natural (constant) birth rate or mortality rate are irrelevant as these variables interact with other variables, such as income and food security. To analyze population dynamics, it is essential to model birth and mortality rates. This study introduces endogenous birth and mortality rates on the basis of some well-known models in the literature of economic growth with population. Fertility is dependent on many factors, such as changes in gender gap in wages (Galor and Weil, 1996), labor market frictions (Adsera, 2005), and age structure (Hock and Weil, 2012). There are many formal models with endogenous fertility (Barro and Becker, 1989; Bosi and Seegmuller, 2012). Becker *et al*. (1990) argue that bringing up children to adulthood is costly. According to Chu *et al.* (2013) the fertility is negatively related to growth rate of human capital. Higher fertility rate has crowding-out effect on household's time endowment and a diluting effect of human capital per member of households. The household takes account of trading off the marginal utility of higher fertility against costs arising from the foregone wages, the dilution of financial assets per capita, as well as the dilution of human capital per capita. Quality-quantity trade-off on children are also emphasized by, for instance, Galor and Weil (1999) and Doepke (2004). As argued by Robinson and Srinivasan (1997), there are close relations between economic development and mortality rate. Endogenous longevity are analyzed by different researchers from economic perspectives (e.g., Blackburn and Cipriani, 2002; Chakraborty, 2004; Hazan and Zoabi, 2006; Bhattacharya and Qiao, 2007, Lancia and Prarolo, 2012). It is demonstrated that life expectancy is influenced by factors such as health care, human capital level and environment (e.g., Schultz 1993, 1998; Boucekkine et al., 2002; Cervellati and Sunde, 2005; Balestra and Dottori (2012). Varvarigos and Zakaria (2013) examine interactions between fertility choice and expenditures on health in the traditional overlapping-generations framework. The pension system has also been emphasized in the literature of longevity (Cigno and Rosati, 1996; Wigger 1999). In Hock and Weil (2012) the interdependence of fertility, the population age structure and intergenerational transfers are analyzed. There are also formal models on longevity, human capital and growth (e.g., Kalemli-Ozcan *et al.*, 2000; Echevarria and Iza, 2006; Heijdra and Romp, 2008; Ludwig and Vogel, 2009; Lee and Mason, 2010; and Ludwig *et al.*, 2012).

This study includes natural resources in the growth model with endogenous population. Long time ago Gordon (1956) emphasized the need for a dynamic approach to resources: "The conservation problem is essentially one which requires a dynamic formulation… The economic justification of conservation is the same as that of any capital investment – by postponing utilization we hope to increase the quantity available for use at a future date. In the fishing industry we may allow our fish to grow and to reproduce so that the stock at a future date will be greater than it would be if we attempted to catch as much as possible at the present time. … [I]t is necessary to arrive at an optimum which is a catch per unit of time, and one must reach this objective through consideration of the interaction between the rate of catch, the dynamics of fish population, and the economic time-preference schedule of the community or the interest rate on invested capital. His is a very complicated problem and I suspect that we will have to look to the mathematical economists for assistance in clarifying it." Although interactions between natural resources and economic growth are dealt with within the neoclassical growth theory in the 1970s (e.g., Plourde, 1970, 1971; Stiglitz, 1974; Clark, 1976; Dasgupta and Heal, 1979), only a few models with both endogenous growth and resources are proposed with endogenous population. Solow (1999) argues for the necessity of taking account of natural resources in the neoclassical growth theory. Nevertheless, Solow does not show how to incorporate possible consumption of renewable resource into the growth model. It is obvious that renewable resources are used both as inputs of production and consumptions of households. But a few growth models with renewable resources consider renewable resource as both input of production and a source of utility (see, Beltratti, *et al*., 1994, Ayong Le Kama, 2001; Eliasson and Turnovsky, 2004; Alvarez-Cuadrado and van Long, 2011). Our model contains the renewable resource as a source of utility and input of production.

Another important feature of our model is endogenous time distribution between work, leisure and children caring. Becker (1965) initiated formally modelling time distribution in theoretical economics. Since then an immense body of empirical and theoretical literature has been published (e.g., Benhabib and Perli, 1994; Ladrón-de-Guevara *et al*. 1997; Jones and Manuelli, 1995; Turnovsky, 1999; Greenwood and Hercowitz, 1991; Rupert *et al*. 1995; Cambell and Ludvigson, 2001). Nevertheless, only a few theoretical economic growth models with renewable resource and population explicitly treat work time as an endogenous variable. This paper introduces endogenous time into the neoclassical growth theory with renewable resource. This paper is to integrate two recent papers by Zhang (2013, 2014). The former paper deals with renewable resource and leisure time in the neoclassical growth theory, while the latter studies dynamic interactions between population and economic growth. This paper integrates the two models to examine dynamic interactions between physical capital, renewable resources and population with endogenous time distribution between leisure, work and children fostering. Our model is also a synthesis of some well-known growth models. The remainder of the paper is organized as follows. Section 2 defines the economic model with endogenous capital accumulation, resource dynamics and population change. Section 3 shows that the motion of the economic system is described by three differential equations and simulates the model. Section 4 carries out comparative dynamics analysis. Section 5 concludes the study.

2. The basic model

The economy has one production sector and one renewable resource. Most aspects of the production sector are similar to the standard one-sector growth model (Zhang, 2005). We refer issues related to modeling management of multiple renewable resource stocks in distinct harvesting grounds and to growth theory with multiple kinds of capital to, for instance, Horan and Shortle (1999), Koundouri and Christou (2006). Households own physical capital of the economy and distribute their incomes to consumption, child bearing, and wealth accumulation. The production sectors or firms use physical capital and labor as inputs. Exchanges take place in perfectly competitive markets. Factor markets work well; factors are inelastically supplied and the available factors are fully utilized at every moment. Saving is undertaken only by households. All earnings of firms are distributed in the form of payments to factors of production. We assume a homogenous population at time $N(t)$. Let $T(t)$ and $\widetilde{T}(t)$ represent for, respectively, the work time and leisure time of a representative household. The total work time is $T(t)N(t)$. The total qualified labor force is

$$\overline{N}(t) \equiv hT(t)N(t), \tag{1}$$

where the parameter h is the human capital.

The labor force is distributed between the two sectors. We select the commodity to serve as numeraire, with all the other prices being measured relative to its price. We assume that wage rates are identical among all professions. The total capital stock of physical capital, $K(t)$, is allocated between the two sectors. We use $N_x(t)$ and $K_x(t)$ to stand for the labor force and capital stocks employed by the resource sector, and $N_i(t)$ and $K_i(t)$ for the labor force and capital stocks employed by the production sector. The assumption of full employment of input factors is represented

$$K_i(t) + K_x(t) = K(t), \quad N_i(t) + N_x(t) = \overline{N}(t). \tag{2}$$

The production sector

We assume that production is to combine labor force, $N_i(t)$, and physical capital, $K_i(t)$, and renewable resource, $K_R(t)$. We use the conventional production function to describe a relationship between inputs and output. Let $F_i(t)$ stand for output level of the production sector at time t. The production function is specified as follows

$$F_i(t) = A_i K_i^{\alpha_i}(t) N_i^{\beta_i}(t) K_R^{\gamma_i}(t), \quad A_i, \alpha_i, \beta_i, \gamma_i > 0, \quad \alpha_i + \beta_i + \gamma_i = 1, \tag{3}$$

where A_i, α_i, β_i and γ_i are positive parameters. Markets are competitive; thus labor and capital earn their marginal products. The rate of interest, $r(t)$, and wage rate, $w(t)$, the price of the resource, $p(t)$, are determined by markets. The marginal conditions are given by

$$r(t) + \delta_k = \frac{\alpha_i F_i(t)}{K_i(t)}, \quad w(t) = \frac{\beta_i F_i(t)}{N_i(t)}, \quad p(t) = \frac{\gamma_i F_i(t)}{K_R(t)}, \tag{4}$$

where δ_k is the given depreciation rate of physical capital.

Change of renewable resources

The modelling of resource and the resource sector follows Zhang (2013). Let $X(t)$ stand for the stock of the resource. The natural growth rate of the resource is assumed to be a logistic function of the existing stock

$$\phi_0 \, X(t)\left(1 - \frac{X(t)}{\phi}\right),$$

where the variable, ϕ, is the maximum possible size for the resource stock, called the carrying capacity of the resource, and the variable, ϕ_0, is "uncongested" or "intrinsic" growth rate of the renewable resource. If the stock is equal to ϕ, then the growth rate should equal zero. If the carrying capacity is much larger than the current stock, then the growth rate per unit of the stock is approximately equal to the intrinsic growth rate. That is, the congestion effect is negligible. The logistic model has been frequently used in the literature of growth with renewable resource (e.g., Brander and Taylor, 1997; Brown, 2000; Hannesson, 2000; Cairns and Tian, 2010, Farmer and Bednar-Friedl, 2011). It should be noted that there are some alternative approaches to renewable resources. For instance, Tornell and Velasco (1992), Long and Wang (2009), and Fujiwara (2011) use linear resource dynamics. In this study, for simplicity we assume both the maximum possible size and the intrinsic growth rate constant. This is a strict assumption as these variables may change due to changes in other conditions (Berck, 1981; Levhari and Withagen, 1992; Benchekroun, 2003, 2008; Wirl, 2004; Jinni, 2006). It should also be mentioned that Munro and Scott (1985), Koskela *et al.* (2002) and Uzawa (2005: Chap. 2) use more general growth functions in their analysis of renewable resources in growth models. Let $F_x(t)$ stand for the harvest rate of the resource. The change rate in the stock is then equal to the natural growth rate minus the harvest rate, that is

$$\dot{X}(t) = \phi_0 \, X(t)\left(1 - \frac{X(t)}{\phi}\right) - F_x(t). \tag{5}$$

We now examine functional form of the harvest rate. We assume a nationally owned open-access renewable resource (Gordon, 1954; Alvarez-Guadrado and Von Long, 2011). With open access, harvesting occurs up to the point at which the current return to a representative entrant equals the entrant's cost. Aside from the stock of the renewable resources, like the good sector there are two factors of production. We use $N_x(t)$ and $K_x(t)$ to stand for the labor force and capital stocks employed by the resource sector. We assume that harvesting of the resource is carried out according to the following harvesting production function

$$F_x(t) = A_x \, X^{b_x}(t) K_x^{\alpha_x}(t) N_x^{\beta_x}(t), \;\; A_x, b_x, \alpha_x, \beta_x > 0, \;\; \alpha_x + \beta_x = 1, \tag{6}$$

where A_x, b_x, α_x and β_x are parameters. The specified form implies that if the capital (like machine) and labour inputs are simultaneously doubled, then harvest is also doubled for a given stock of the resource at a given time. It should be noted that the Schaefer harvesting production function which is taken on the following form (Schaefer, 1957):

$$F_x(t) = A_x X(t) N_x(t),$$

is evidently a special case of (6). The Schaefer production function does not take account of capital (or with capital being fixed). Firms choose the capital and labor inputs in harvesting. The marginal conditions are given as follows

$$r(t) + \delta_k = \frac{\alpha_x \, p(t) F_x(t)}{K_x(t)}, \quad w(t) = \frac{\beta_x \, p(t) F_x(t)}{N_x(t)}. \tag{7}$$

Consumer behaviors

Consumers decide time distribution, consumption level of commodity, number of children, and amount of saving. Different from the optimal growth theory in which utility defined over future consumption streams is used, we use an alternative approach to household proposed by Zhang (1993). To describe behavior of consumers, we denote per family wealth by $\bar{k}(t),$ where $\bar{k}(t) = K(t)/N(t).$ Per family current income from the interest payment and the wage payments is

$$y(t) = r(t)\bar{k}(t) + hT(t)w(t).$$

We call $y(t)$ the current income in the sense that it comes from consumers' payment for efforts and consumers' current earnings from ownership of wealth. The total value of wealth that a family can sell to purchase goods and to save is equal to $\bar{k}(t)$. Here, we assume that selling and buying wealth can be conducted instantaneously without any transaction cost. The disposable income per family is given by

$$\hat{y}(t) = y(t) + \bar{k}(t). \tag{8}$$

Let $n(t)$ and $p_b(t)$ stand for the birth rate and the cost of birth. There are many factors which may affect costs of bringing up children. Following Zhang (2014), we assume that children will have the same level of wealth as that of the parent. Hence, in addition to the time spent on children, the cost of the parent is given by

$$p_b(t) = n(t)\bar{k}(t). \tag{9}$$

Here, we neglect other costs such as purchases of goods and services. In the fertility choice model by Barro and Becker (1989), the cost also includes consumption of goods. We now introduce time spent on children. Becker (1981) emphasizes costs of the mother's time on rearing children to adulthood. We consider the following relation between birth rate and the parent's time on raising children

$$\bar{T}(t) = \theta n(t), \quad \theta \geq 0. \tag{10}$$

The specified function form implies that if the parents want more children, they spend more time on child caring. This requirement is strict as child caring tends to exhibit increasing return to scale. The time required to take care of per child tends to fall as the family has more children. We require the constant return to scale because this assumption makes the analysis mathematically tractable.

The household distributes the total available budget between saving, $s(t)$, consumption of goods, $c(t)$, consumption of the resource good, $c_x(t)$, and bearing children, $p_b(t)$. The budget constraint is

$$c(t) + s(t) + p(t)c_x(t) + \bar{k}(t)n(t) = \hat{y}(t). \tag{11}$$

It should be noted that we omit possible issues related to wealth heritage. It is conceptually not difficult to take account of wealth heritage within our framework as we can include heritage in the disposable income. From the appendix it can be seen that we can analyze the resulted problem by properly specifying death rate function.

We consider that except work and child fostering, parents also have their leisure. An adult is faced with the following time constraint

$$T(t) + \overline{T}(t) + \widetilde{T}(t) = T_0, \tag{12}$$

where T_0 is the total available time for leisure, work and children caring. Substituting (12) into (11) yields

$$c(t) + s(t) + p(t)c_x(t) + \bar{k}(t)n(t) + h\overline{T}(t)w(t) + h\widetilde{T}(t)w(t) = \bar{y}(t), \tag{13}$$

where

$$\bar{y}(t) \equiv (1 + r(t))\bar{k}(t) + h\,w(t)T_0.$$

The right-hand side is the "potential" income that the family can obtain by spending all the available time on work. The left-hand side is the sum of the consumption cost, the saving, the opportunity cost of bearing children, and opportunity cost of leisure. Insert (10) in (13)

$$c(t) + s(t) + p(t)c_x(t) + \tilde{w}(t)n(t) + h\widetilde{T}(t)w(t) = \bar{y}(t), \tag{14}$$

where

$$\tilde{w}(t) \equiv \bar{k}(t) + h\theta\,w(t).$$

The variable $\tilde{w}(t)$ is the opportunity cost of children fostering. Like Barro and Becker (1989), we assume that the parents' utility is dependent on the number of children. We assume that the utility is dependent on $c(t)$, $s(t)$, $\widetilde{T}(t)$, and $n(t)$ as follows

$$U(t) = c^{\xi_0}(t)s^{\lambda_0}(t)\widetilde{T}^{\sigma_0}(t)n^{\upsilon_0}(t)c_x^{\chi_0}(t),$$

where ξ_0 is called the propensity to consume, λ_0 the propensity to own wealth, σ_0 the propensity to use leisure time, υ_0 the propensity to have children, and χ_0 the propensity to consume renewable resource. It should be noted that in this study we assume that the parents' utility is dependent only on the number of their children. According to Soares (2005) parents' utility depends not only on their surviving offspring, but also on length of each surviving child's lifespan. The first-order condition of maximizing $U(t)$ subject to (14) yields

$$c(t) = \xi\,\bar{y}(t), \;\; s(t) = \lambda\,\bar{y}(t), \;\; \tilde{T}(t) = \frac{\sigma\,\bar{y}(t)}{w(t)}, \;\; c_x(t) = \frac{\chi\,\bar{y}(t)}{p(t)}, \;\; n(t) = \frac{\upsilon\,\bar{y}(t)}{\tilde{w}(t)}, \qquad (15)$$

where

$$\sigma \equiv \frac{\rho\sigma_0}{h}, \;\; \xi \equiv \rho\xi_0, \;\; \lambda \equiv \rho\lambda_0, \;\; \chi \equiv \rho\chi_0, \;\; \upsilon \equiv \rho\upsilon_0, \;\; \rho \equiv \frac{1}{\sigma_0 + \xi_0 + \lambda_0 + \chi_0 + \upsilon_0}.$$

The birth and mortality rates and population dynamics

To illustrate our approach and show how our model is related to the literature, we consider the Haavelmo model (Haavelmo, 1954; Stutzer, 1980)

$$\dot{N}(t) = N(t)\left(a - \frac{\beta N(t)}{Y(t)}\right), \;\; a, \beta > 0, \quad Y(t) = AN^{\alpha}(t), \;\; A > 0, \;\; 0 < \alpha < 1,$$

where $N(t)$ is the population, $Y(t)$ is real output, and a, β, α and A are parameters. Insert $Y(t) = AN^{\alpha}(t)$ in the differential equation

$$\frac{\dot{N}(t)}{N(t)} = a - \frac{\beta}{f(t)} = a - \frac{\beta N^{1-\alpha}(t)}{A},$$

where $f \equiv Y/N$ is per capita output. In the Haavelmo model, there is no physical capital accumulation. As the change rate in the population is the birth rate minus the mortality rate, we may interpret that in the Haavelmo model the birth rate ($= a$) is constant and the mortality rate ($= \beta / f(t)$) is negatively related to per capita income.

Another typical approach in the literature of population growth and economic development is the so-called Ramsey model. An example is the model by Chu *et al.* (2013). The household decides the fertility rate by maximizing the discounted sum of per capita utility across subject to the asset accumulation

$$U = \int_0^{\infty} u(c(t), n(t))e^{-\rho t}\,dt,$$
$$\text{s.t.: } \dot{a}(t) = (r(t) - n(t))a(t) + w(t)l(t) - c(t),$$

where $c(t)$ is the per capita consumption of final goods at time t, $n(t)$ is the number of births per person, $a(t)$ is the amount of financial assets per capita, $r(t)$ is the rate of return on assets, $w(t)$ is the wage rate, and $l(t)$ is human capital-embodied labor supply. The total population growth is $\dot{N} = nN$. As mortality is assumed to be zero in this model, n is also the growth rate of the population. The type of population growth is also accepted, for instance, by Razin and Ben-Zion (1975) and Yip and Zhang (1997).

According to the definitions, the population change follows

$$\dot{N}(t) = (n(t) - d(t))N(t), \tag{16}$$

where $n(t)$ and $d(t)$ are respectively the birth rate and mortality rate. It should be noted that Tournemaine and Luangaram (2012) use the following technology of production of children: $n(t) = bT_b^{\theta}(t)$, where $T_b(t)$ is the time of rearing children and b and θ are parameters. They consider mortality rate constant. In the Haavelmo model, the mortality rate is negatively related to per capita income. In this study we assume that the mortality rate is negatively related to the disposable income in the following way

$$d(t) = \frac{\bar{\upsilon}\, N^b(t)}{\bar{y}^a(t)}, \tag{17}$$

where $\bar{\upsilon} \geq 0$, $a \geq 0$. We call $\bar{\upsilon}$ the mortality rate parameter. As in the Haavelmo model, an improvement in living conditions implies that people live longer. The term $N^b(t)$ takes account of possible influences of the population on mortality. For instance, when the population is increased, environment tends to be deteriorated. We may take account of this kind of environmental effects by the term. In this case, it is reasonable require b to be positive. It should be noted that the sign of the parameter is generally ambiguous in the sense that the population may also have a positive impact on mortality. Insert (10) and (13) in (12)

$$\dot{N}(t) = \left(\frac{\upsilon\, \bar{y}(t)}{\tilde{w}(t)} - \frac{\bar{\upsilon}\, N^b(t)}{\bar{y}^a(t)}\right) N(t). \tag{18}$$

The equation describes the population dynamics. It should be noted that to properly describe the population change, we should also model the age structure. Like in most of the models in continuous time in the literature of economic growth and population change, for simplicity of analysis we don't deal with complicated issues related to the age structure in this stage of research.

Wealth dynamics

We now find dynamics of wealth accumulation. According to the definition of $s(t)$, the change in the household's wealth is given by

$$\dot{\bar{k}}(t) = s(t) - \bar{k}(t) = \lambda \bar{y}(t) - \bar{k}(t). \tag{19}$$

The equation simply says that the change in wealth is equal to saving minus dissaving.

Demand for and supply of goods and resource

The national saving is the sum of the households' saving. As output of the capital goods sector is equal to the net savings and the depreciation of capital stock, we have

$$S(t) + C(t) + n(t)\bar{k}(t)N(t) - K(t) + \delta_k\, K(t) = F_i(t), \tag{20}$$

where $S(t) - K(t) + \delta_k K(t)$ is the sum of the net saving and depreciation, $C(t) + n(t)\bar{k}(t)N(t)$ is the total expenditure and

$$S(t) = s(t)N(t), \quad C(t) = c(t)N(t), \quad K(t) = \bar{k}(t)N(t).$$

The demand for and supply of the resource balance at any point of time

$$c_x(t)N(t) + K_R(t) = F_x(t). \tag{21}$$

We have thus built the dynamic model. It should be noted that the model is general in the sense that the Solow model and the Haavelmo model can be considered as special cases of our model. Moreover, as our model is based on the some well-known mathematical models and includes some features which no other single theoretical model explains, we should be able to explain some interactions which other formal models fail to explain. We now examine dynamics of the model.

3. The dynamics and its properties

We built the growth model with endogenous wealth, population and resources. The model contains many variables and these variables are interrelated to each other in complicated ways. It is difficult to get analytical properties of the nonlinear differential equations. For illustration, we simulate the model to demonstrate how variables change over time. First, we introduce

$$z(t) \equiv \frac{r(t) + \delta_k}{w(t)}.$$

We show that the dynamics can be expressed by a three-dimensional differential equations system with $z(t)$, $N(t)$, and $X(t)$ as the variables.

Lemma

The dynamics of the economic system is governed by the three dimensional differential equations

$$\begin{aligned} \dot{z}(t) &= \tilde{\Omega}_z(z(t), N(t), X(t)), \\ \dot{N}(t) &= \tilde{\Omega}_N(z(t), N(t), X(t)), \\ \dot{X}(t) &= \tilde{\Omega}_X(z(t), N(t), X(t)), \end{aligned} \tag{22}$$

where the functions $\tilde{\Omega}_z$, $\tilde{\Omega}_N$, and $\tilde{\Omega}_X$ are functions of $z(t)$, $N(t)$, and $X(t)$ defined in the Appendix. Moreover, all the other variables are determined as functions of $z(t)$, $N(t)$, and $X(t)$ at any point in time by the following procedure: $\bar{k}(t)$ by (A24) $\rightarrow N_i(t)$ and $N_x(t)$ by (A15) $\rightarrow$ $r(t)$, $w(t)$ and $p(t)$ by (A6) $\rightarrow$ $K_i(t)$ and $K_x(t)$ by (A1) $\rightarrow$ $\bar{y} = (1 + r(t))\bar{k}(t) + hT_0 w(t)$ $\rightarrow c(t)$, $s(t)$, $\tilde{T}(t)$, $c_x(t)$, and $n(t)$ by (14) $\rightarrow K_R(t)$ by (A3) $\rightarrow \bar{T}(t)$ by (A10) $\rightarrow T(t)$ by (A12) $\rightarrow$ $\bar{N}(t)$ by (1) $\rightarrow$ $F_i(t)$ by (3) $\rightarrow$ $F_x(t)$ by (6).

The differential equations system (22) contains three variables, $z(t)$, $N(t)$, and $X(t)$. As the expressions are too complicated, we simulate the model to illustrate behavior of the system. In the remainder of this study, we specify the depreciation rates by $\delta_k = 0.05$. The available time is $T_0 = 24$. The depreciation rate of physical capital is often fixed around 0.05 in economic studies. We specify the other parameters as follows

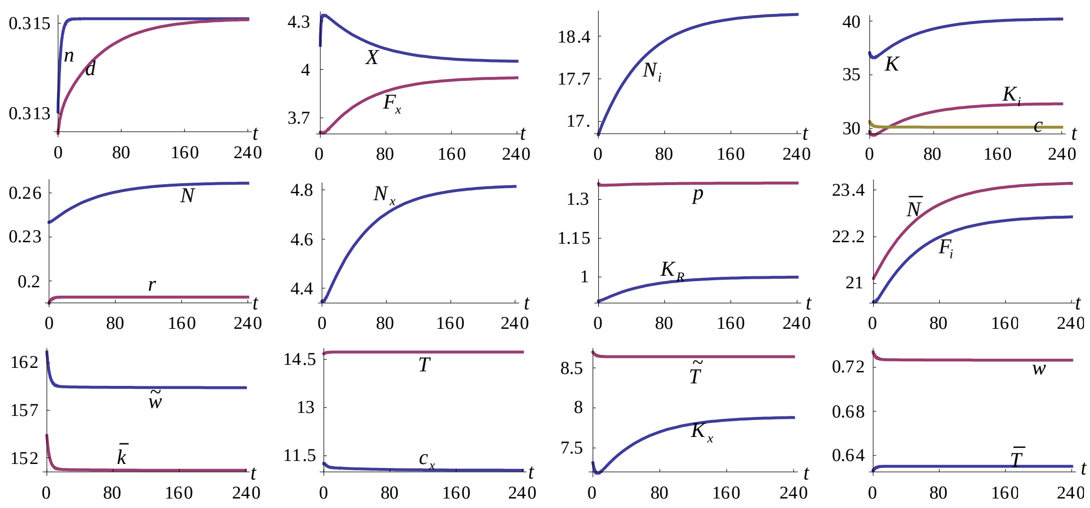
$$\alpha_i = 0.34,\ \beta_i = 0.6,\ \alpha_x = 0.35,\ \lambda_0 = 0.6,\ \xi_0 = 0.08,\ \chi_0 = 0.06,\ \sigma_0 = 0.16,$$
$$\upsilon_0 = 0.2,\ A_i = 1.2,\ A_x = 0.6,\ \phi = 3,\ \phi_0 = 6,\ b_x = 0.1,\ \theta = 2,\ h = 6,$$
$$a = 0.1,\ b = 0.06,\ \bar{\upsilon} = 0.6. \tag{23}$$

The propensity to save is 0.6 and the propensity to consume goods is 0.08. The propensity to have children is 0.2. The technological parameters of the two sectors are specified at $A_i = 1.2$ and $A_x = 0.6$. The human capital level is 6. We specify initial conditions as follows

$$z(0) = 0.32,\ N(0) = 0.24,\ X(0) = 4.15.$$

The simulation result is plotted in Figure 1. From the plots we see that both the birth rate and mortality rate rise from their initial values. As the birth rate grows faster than the mortality till they achieve their equilibrium value, the population expands over time. The resource stock rises and soon falls. The output levels, labor and capital inputs of the two sectors are increased. The resource input of the production sector is increased. The time distribution is slightly changed. The total labor input is increased. The opportunity cost of children fostering, wealth, consumption of goods and resource per household are reduced. The price of resource is slightly increased and the wage rate is reduced slightly.

Figure 1 - The Motion of the Economic System near a Stable Equilibrium Point



The simulation demonstrates that the dynamic system has two equilibrium points. Figure 1 identifies the equilibrium point with higher equilibrium values of renewable resources. We list the equilibrium values of the variables as follows

$$N = 0.267,\ K = 40.23,\ X = 4.05,\ \overline{N} = 23.6,\ N_i = 18.78,\ N_x = 4.82,\ K_i = 32.34,$$
$$K_x = 7.89,\ K_R = 1,\ F_i = 22.74,\ F_x = 3.95,\ n = d = 0.31,\ r = 0.189,\ w = 0.73,$$
$$\tilde{w} = 159.39,\ p = 1.36,\ \bar{k} = 150.67,\ T = 14.72,\ \tilde{T} = 8.64,\ \overline{T} = 0.63,\ c = 30.13,$$
$$c_x = 11.05. \tag{24}$$

The equilibrium point in (24) confirms the one in Figure 1. We calculate the three eigenvalues at this equilibrium point as follows

$$\{-1.14,\ -0.31,\ -0.021\}.$$

As the three eigenvalues are negative, the unique equilibrium point is locally stable. Hence, the system always approaches its equilibrium if it is not far from the equilibrium point. We will conduct comparative dynamic analysis near this equilibrium point.

We mentioned the existence of two equilibrium points. The equilibrium values of the other equilibrium point are listed as follows

$$N = 0.263,\ K = 39.32,\ X = 1.62,\ \overline{N} = 23.26,\ N_i = 18.51,\ N_x = 4.75,\ K_i = 31.61,$$
$$K_x = 7.71,\ K_R = 897,\ F_i = 22.22,\ F_x = 3.54,\ n = d = 0.32,\ r = 0.189,\ w = 0.72,$$
$$\tilde{w} = 159.02,\ p = 1.49,\ \bar{k} = 149.38,\ T = 14.73,\ \tilde{T} = 8.64,\ \overline{T} = 0.63,\ c = 29.87,$$
$$c_x = 11.05.$$

We calculate the three eigenvalues at the equilibrium point

$$\{1.16,\ -0.31,\ -0.018\}.$$

One of the eigenvalues is real and positive. Accordingly the equilibrium point is unstable. We simulated the model a few times with initial conditions very near to the equilibrium point. The system moves away from the unstable equilibrium point and moves toward the other stable equilibrium point.

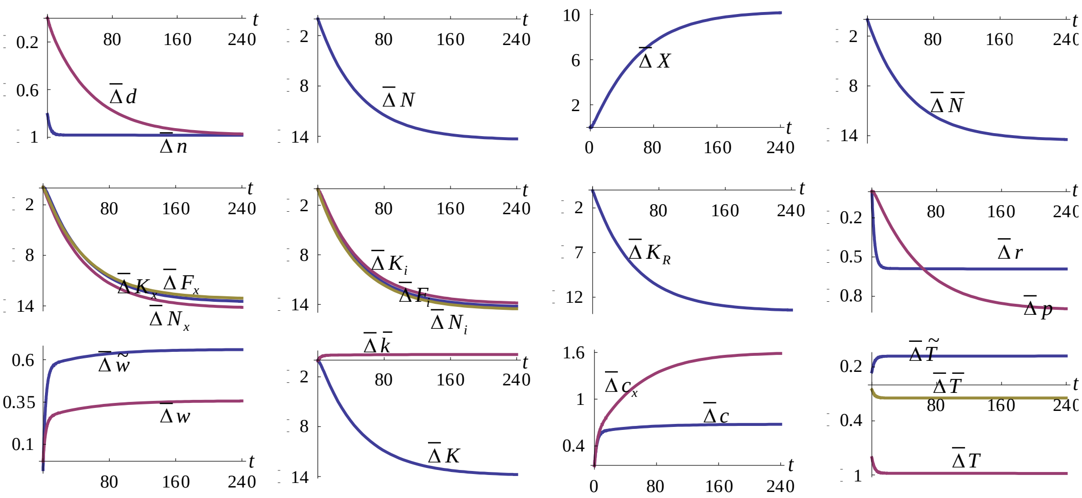
4. Comparative dynamic analysis in some parameters by simulation

Tournemaine and Luangaram (2012: 925) point out "depending on the country, population growth may contribute, deter, or even have no impact on economic development. This ambiguous result is explained by the fact that the effects of population growth change over time. For example, a higher fertility rate can have a short-term negative effect caused by the cost of expenditures on children whereas it has a long-run positive effect through the larger labor force it generates." We now examine impact of changes in some parameters on dynamic processes of the system. We use a variable $\overline{\Delta} x(t)$ to stand for the change rate of the variable, $x(t)$, in percentage due to changes in the parameter value. In order to examine how each variable is affected over time, we should follow the motion of the entire system as each variable is related to the others in the dynamic system.

The propensity to have children being reduced

We allow the propensity to have children to be reduced as follows: $\upsilon_0 : 0.2 \Rightarrow 0.198$. The simulation results are plotted in Figure 2. When people reduce their propensity to have children, the birth rate is lowered. The population and work time are reduced. The net result of the fall in the population and the reduction in leisure time rises the total labor supply. The total capital is reduced in association with the fall in the population and the rise in wealth per household. The wage rate and opportunity cost of children fostering are enhanced. The output levels, labor inputs, and capital inputs of the two sectors are all reduced. The rate of interest and price of the resource are reduced. The consumption levels of goods and resources are increased. The household works less hours, increases leisure hours, and spends less time on children fostering. It should be noted that the model by Varvarigos and Zakaria (2013) predicts a fertility decline along the process of economic growth. The two-period overlapping-generations model by Strulik (2008) predicts an inverted U-shape relationship between fertility and income. Acemoglu and Johnson (2007) try to analyze the impact of life expectancy on economic growth. There is no evidence of a positive effect identified. It is also observed that there is a decline of the fertility rate alone the process of economic development (e.g., Kirk, 1996; Ehrlich and Lui 1997; Galor, 2012). Although this study accepts an analytical framework different from the traditional models, our analyses show that change directions of the birth rate and mortality rate are situation-dependent.

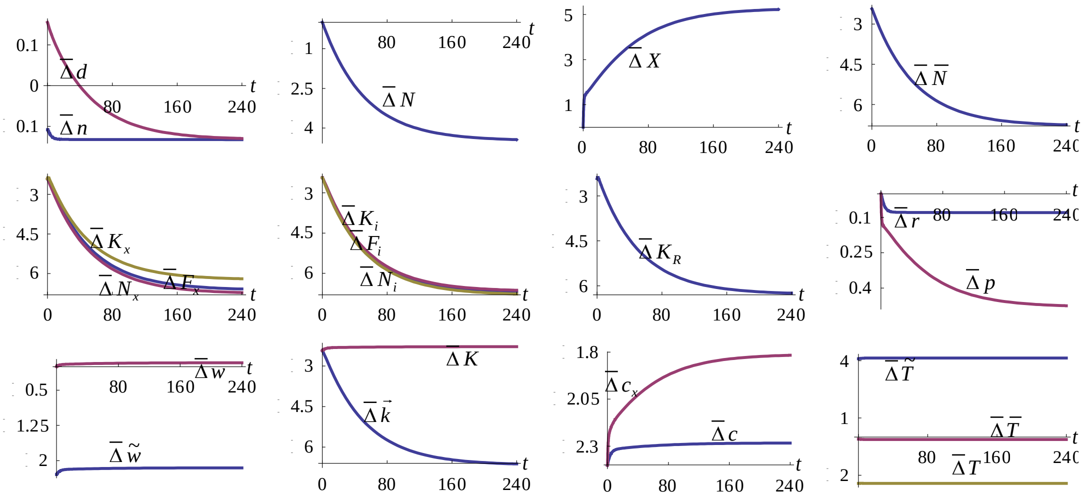
Figure 2 - The Propensity to Have Children Being Reduced



The propensity to consume leisure being increased

There is a strong desire for enjoying free time for many people in modern times. It is important to see how a change in the time distribution propensity has effects on birth rates and economic development. We allow the propensity to have leisure time to be increased as follows: $\sigma_0 : 0.15 \Rightarrow 0.16$. The simulation results are plotted in Figure 3. The change in the propensity enhances the leisure time, lowers the work time, and has slight impact on the time of children fostering. As more time is spent on leisure, the total labor supply is reduced. The wage rate is slightly affected. The net result of these two changes implies a falling in wage income. As the wage income and wealth are reduced, the birth rate falls. Initially the mortality rate rises. But in association with falling birth rate, mortality rate falls late on. The total population is reduced. Both the rate of interest and price of resources are reduced. The resource stock is augmented. The output levels, labor inputs, and capital inputs of the two sectors are all reduced. The production sector also uses less resource. The consumption levels of goods and resources are reduced.

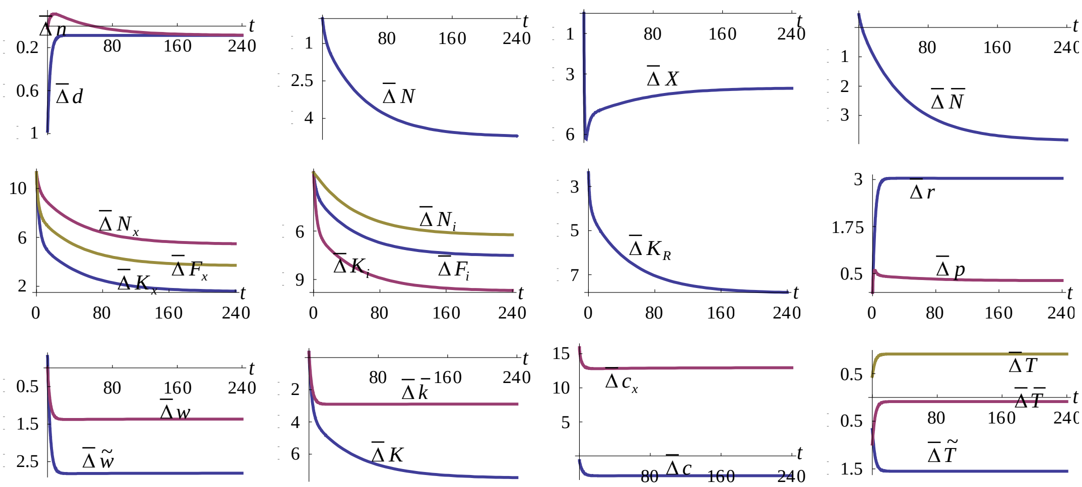
Figure 3 - The Propensity to Consume Leisure Being Increased



The propensity to consume resources being increased

We study what will happen to the system when the propensity to consume resources is increased as follows: $\chi_0 : 0.06 \Rightarrow 0.07$. The simulation results are plotted in Figure 4. The change in the propensity enhances the leisure time, lowers the work time, and has slight impact on the time of children fostering. As more time is spent on leisure, the total labor supply is reduced. The wage rate is slightly affected. The net result of these two changes implies a falling in wage income. As the wage income and wealth are reduced, the birth rate falls. Initially the mortality rate rises. But in association with falling birth rate, mortality rate falls late on. The total population is reduced. Both the rate of interest and price of resources are reduced. The resource stock is augmented. The output levels, labor inputs, and capital inputs of the two sectors are all reduced. The production sector also uses less resources. The consumption levels of goods and resources are reduced.

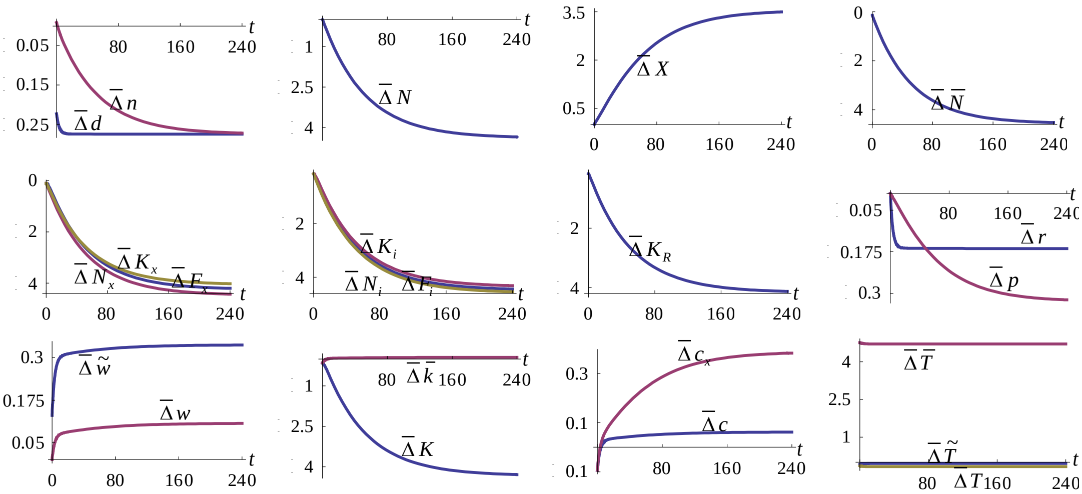
Figure 4 - The Propensity to Consume Resources Being Increased



More time required for children fostering

Another important issue is change in time required for bringing up children. We now examine what will happen if the time required to take care of children is increased as follows: $\theta: 2 \Rightarrow 2.1$. The simulation results are plotted in Figure 5. In terms of opportunity costs it is more expensive to have children. The birth rate falls. The population and mortality rate are also reduced. As the household spends more time on children fostering. Both the leisure time and work time are slightly reduced. Wealth per household is slightly changed and wage rate is increased. The opportunity cost of children fostering is increased. The fall in birth rate causes the population to fall and the falling population reduces the mortality rate. The dynamic interactions between the birth rate, mortality rate, and population continue until the birth rate and mortality rate becomes equal. The total labor force falls as the population falls and the work time is almost not affected. The population reduction is also associated with falling in the national capital stock. Both the rate of interest and price of resources are reduced. The resource stock is augmented. The output levels, labor inputs, and capital inputs of the two sectors are all reduced. The production sector also uses less resources. The consumption levels of goods and resources are slightly reduced initially and then increased.

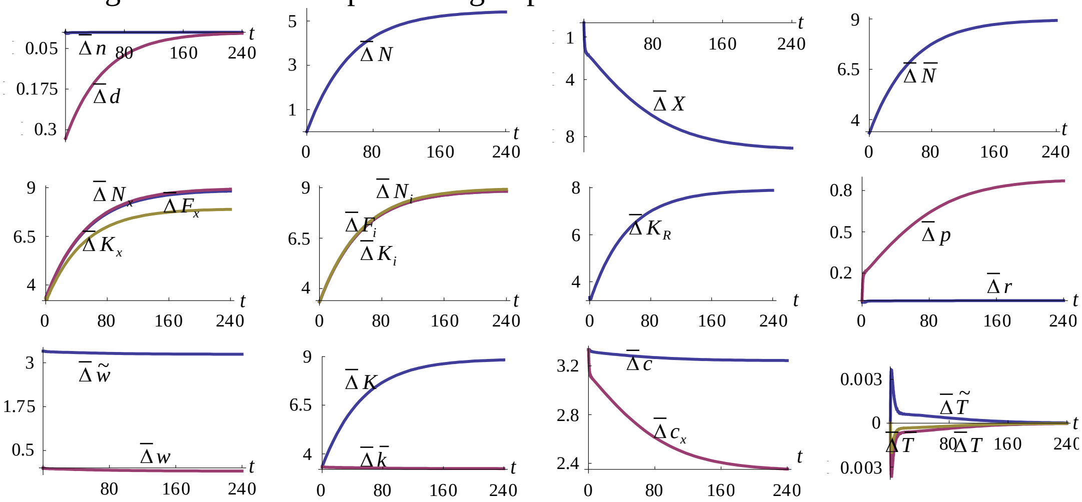
Figure 5 - More Time Required for Children Fostering



Human capital being improved

Relations between human capital and population growth have caused much attention in public as well as academic discussions. Although this study does not take account of endogenous human capital and education cost, we are interested in what will happen if human capital is improved as follows: $h: 6 \Rightarrow 6.2$. The simulation results are plotted in Figure 6. The human capital improvement has light effects on the time distribution. The wage income hTw is increased, even though the wage rate per qualified unit time w falls slightly in association with rising in capital. The consumption levels of goods and resources and wealth per household are enhanced in association in rise in the wage income. The opportunity cost pf children fostering is increased. The birth rate is almost not affected and the mortality rate is reduced. Hence the population is expanded. The total labor force is augmented as the population rises and the work time is almost not affected. The population expansion is also associated with rises in the national capital stock. The rate of interest is slightly affected and price of resources is increased. The resource stock is lowered. The output levels, labor inputs, and capital inputs of the two sectors are all increased. The production sector also uses more resources.

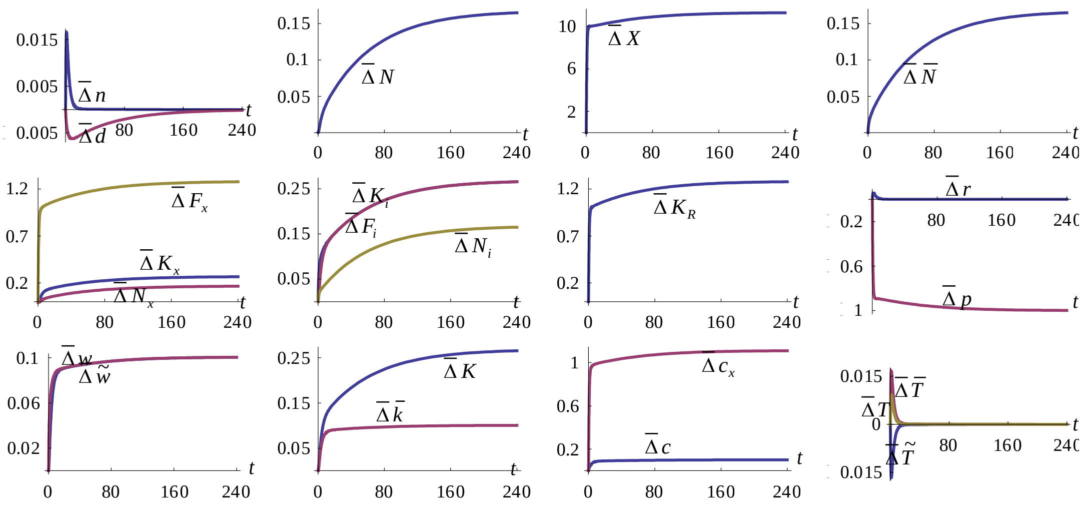
Figure 6 - Human Capital Being Improved



The resource capacity being expanded

We now demonstrate what will happen if the resource capacity is increased as follows: $\phi : 6 \Rightarrow 6.4$. The simulation results are plotted in Figure 7. The resource stock, resource input of the production sector, and consumption level of resource are all increased, while the resource price is reduced. In association with rises in the resource stock the birth rate is slightly increased and the mortality rate is slightly reduced. The population is hence augmented. The time distribution is slightly affected. The total labor input is also increased. The output levels, labor inputs, and capital inputs of the two sectors are all increased. The rate of interest is slightly affected. The wage rate and the opportunity cost pf children fostering are increased.

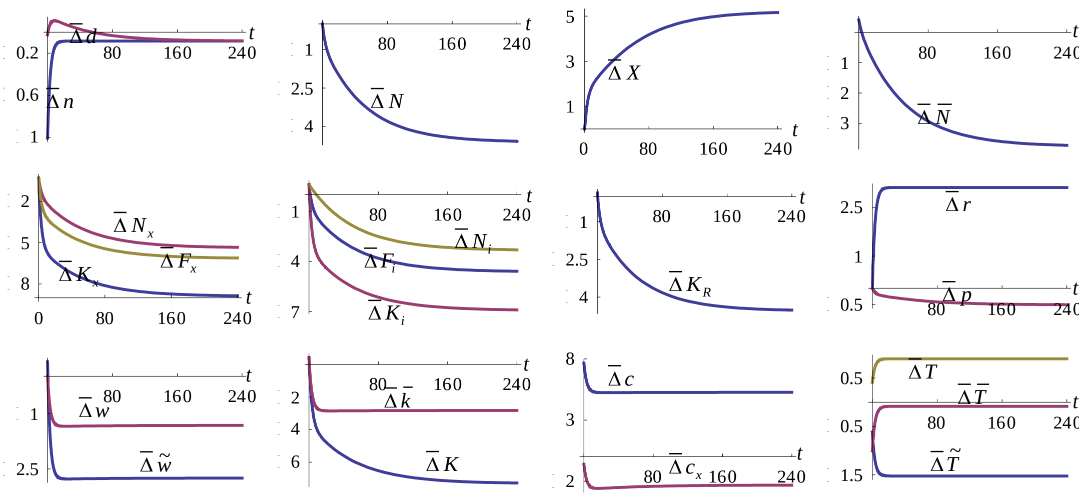
Figure 7 - The Resource Capacity Being Expanded



A rise in the propensity to consume

There are a few formal growth models with microeconomic foundation which explicitly deal with relations between the propensity to consume and population growth. The following simulation shows that changes in the propensity to consume may strongly affect the population dynamics. We now increase the propensity to consume as follows: $\xi_0 : 6 \Rightarrow 6.2$. The simulation results are plotted in Figure 8. The consumption level of goods is increased. The work time is increased, while the time for children fostering and leisure are reduced. As more weight is for consuming, the household has less interest in fostering children. Although the opportunity cost of children caring is lowered, the birth rate is reduced. On the other hand, the mortality rate is slightly increased. The population is thus reduced. The total labor supply is increased. As the propensity to consume is increased, the wealth per household and total capital stock are reduced. The output levels, labor inputs, and capital inputs of the two sectors are all reduced. The production sector also uses less resource. The rate of interest is increased and price of resources is reduced. The resource stock is expanded.

Figure 8 - A Rise in the Propensity to Consume



5. Concluding Remarks

This paper dealt with dynamic interdependence between population change, wealth accumulation, and resource dynamics with endogenous time distribution among work, leisure and children fostering. Our model synthesized economic mechanisms of some well-known models within a compact framework. The wealth accumulation and economic production were built on the Solow growth model. The dynamics of birth and mortality rates were influenced by the Haavelmo population model and the Barro-Becker fertility choice model. The resource change was modeled on the basis of some growth models with renewable resources. The time distribution between work time, leisure time and time of children fostering was influenced by approaches of neoclassical economics. We were able to synthesize these dynamic forces in a compact framework by applying an alternative utility function proposed by Zhang. The two-sector model describes a dynamic interdependence between population change, wealth accumulation, and resource dynamics with endogenous time distribution in a perfectly competitive market. We simulated the model to demonstrate existence of equilibrium points and motion of the dynamic system. We also examined the effects of changes on the motion of the economic system in the propensity to have children, the propensity to consume goods, the propensity to consume resources, the resource capacity, human capital, and time required for children fostering.

Appendix: Proving the Lemma

We now confirm the three-dimensional differential equations in lemma. From (4) and (7), we obtain

$$z \equiv \frac{r+\delta_k}{w} = \frac{\tilde{\alpha}_i N_i}{K_i} = \frac{\tilde{\alpha}_x N_x}{K_x}, \tag{A1}$$

where

$$\tilde{\alpha}_j \equiv \frac{\alpha_j}{\beta_j}, \quad j = i, x.$$

From (4) and (7) we solve

$$K_R = \frac{\gamma_i \beta_x F_x N_i}{\beta_i N_x}. \tag{A2}$$

Insert (6) in (A2)

$$K_R = \varphi_x N_i, \tag{A3}$$

where we also use (A1) and

$$\varphi_x(z, X) \equiv \frac{\gamma_i \beta_x \tilde{\alpha}_x^{\alpha_x} A_x X^{b_x}}{\beta_i z^{\alpha_x}}.$$

Insert (A3) in (3)

$$F_i = A_i \varphi_x^{\gamma_i} K_i^{\alpha_i} N_i^{\beta_i+\gamma_i}. \tag{A4}$$

Substituting (A4) into (4) yields

$$r + \delta_k = \frac{\alpha_i A_i \varphi_x^{\gamma_i} N_i^{\beta_i+\gamma_i}}{K_i^{\beta_i+\gamma_i}}, \quad w = \frac{\beta_i A_i \varphi_x^{\gamma_i} K_i^{\alpha_i}}{N_i^{\alpha_i}}, \quad p = \frac{\gamma_i A_i \varphi_x^{\gamma_i-1} K_i^{\alpha_i}}{N_i^{\alpha_i}}. \tag{A5}$$

Insert (A1) in (A5)

$$r = \frac{\alpha_i A_i \varphi_x^{\gamma_i} z^{\beta_i+\gamma_i}}{\tilde{\alpha}_i^{\beta_i+\gamma_i}} - \delta_k, \quad w = \frac{\beta_i A_i \tilde{\alpha}_i^{\alpha_i} \varphi_x^{\gamma_i}}{z^{\alpha_i}}, \quad p = \frac{\gamma_i A_i \tilde{\alpha}_i^{\alpha_i} \varphi_x^{\gamma_i-1}}{z^{\alpha_i}}. \tag{A6}$$

We thus determine p, r, and w as functions of z and X.

Insert $c_x = \chi \bar{y} / p$ from (15) in (21)

$$\frac{\chi \bar{y} N}{p} + K_R = F_x . \tag{A7}$$

Substituting $w = \beta_x \, p F_x / N_x$ and (A3) into (A7) yields

$$\bar{y} N + \frac{p \varphi_x}{\chi} N_i = \frac{w N_x}{\chi \beta_x}. \tag{A8}$$

Insert $\bar{y} = (1 + r)\bar{k} + h w T_0$ in (A8)

$$(1 + r)K + h w T_0 N + \frac{p \varphi_x}{\chi} N_i = \frac{w N_x}{\chi \beta_x}, \tag{A9}$$

where we use $K = \bar{k} N$. From (1) and (12), we have

$$\bar{N} = \left(T_0 - \theta n - \tilde{T}\right) h N, \tag{A10}$$

where we use $\bar{T} = \theta n$. Insert (15) in (A10)

$$\bar{N} = \left(T_0 - \left(\frac{\theta \upsilon}{\tilde{w}} + \frac{\sigma}{w} \right) \bar{y} \right) h N. \tag{A11}$$

Insert (A1) in (2)

$$\tilde{\alpha}_i N_i + \tilde{\alpha}_x N_x = z K, \quad N_i + N_x = \bar{N}. \tag{A12}$$

Insert (A12) in (A9)

$$\varphi_1 N_i + \varphi_2 \; N_x = \varphi_0 , \tag{A13}$$

where

$$\varphi_1(z, X, N) \equiv - \left(\frac{(1 + r)\tilde{\alpha}_i}{z} + \frac{p \varphi_x}{\chi} \right), \quad \varphi_2(z, X, N) \equiv \frac{w}{\chi \beta_x} - \frac{(1 + r)\tilde{\alpha}_x}{z},$$
$$\varphi_0(z, X, N) \equiv h w T_0 N.$$

From (A11) and (A12), we have

$$N_i + N_x = \varphi_k\left(\bar{k}, z, X, N\right) \equiv \left(T_0 - \left(\frac{\theta \upsilon}{\tilde{w}} + \frac{\sigma}{w} \right) \bar{y} \right) h N. \tag{A14}$$

Solve (A13) and (A14) with N_i and N_x as variables

$$N_i = \frac{\varphi_0 - \varphi_2 \varphi_k}{\varphi_1 - \varphi_2}, \quad N_x = \frac{\varphi_k \varphi_1 - \varphi_0}{\varphi_1 - \varphi_2}. \tag{A15}$$

From (20) we have

$$s N + c N + n \bar{k} N - \delta \bar{k} N = F_i , \tag{A16}$$

where $\delta \equiv 1 - \delta$. Insert $n = \upsilon \bar{y} / \tilde{w}$ (A4) and (15) in (A16)

$$(\lambda + \xi) \bar{y} N + \frac{\upsilon \bar{k} N \bar{y}}{\tilde{w}} - \delta \bar{k} N = A_i \varphi_x^{\gamma_i} K_i^{\alpha_i} N_i^{\beta_i + \gamma_i} . \tag{A17}$$

Insert $\bar{y} = (1 + r)\bar{k} + h w T_0$ in (A17)

$$\bar{r} \bar{k} + \frac{\upsilon \bar{k} \bar{y}}{\tilde{w}} + (\lambda + \xi) h T_0 w = \frac{A_i \varphi_x^{\gamma_i} K_i^{\alpha_i} N_i^{\beta_i + \gamma_i}}{N}, \tag{A18}$$

where

$$\bar{r} \equiv (\lambda + \xi)(1 + r) - \delta .$$

Substituting (A1) into (A18) yields

$$\bar{r} \bar{k} + \frac{\upsilon \bar{k} \bar{y}}{\tilde{w}} + (\lambda + \xi) h T_0 w = \tilde{\varphi} N_i , \tag{A19}$$

where

$$\tilde{\varphi}(z, X, N) \equiv \frac{A_i \varphi_x^{\gamma_i} \tilde{\alpha}_i^{\alpha_i}}{z^{\alpha_i} N} .$$

Insert (A15) in (A19)

$$\bar{r} \bar{k} + \frac{\upsilon \bar{k} \bar{y}}{\tilde{w}} + \omega_1 \varphi_k = \omega_0 , \tag{A20}$$

where

$$\omega_1(z, X, N) \equiv \frac{\tilde{\varphi} \varphi_2}{\varphi_1 - \varphi_2}, \quad \omega_0(z, X, N) \equiv \frac{\tilde{\varphi} \varphi_0}{\varphi_1 - \varphi_2} - (\lambda + \xi) h T_0 w .$$

Insert the definition of φ_k in (A20)

$$\frac{\bar{r} \bar{k}}{\omega_1 h N} + \left(\frac{\bar{k}}{\omega_1 h N} - \theta \right) \frac{\upsilon \bar{y}}{\tilde{w}} + T_0 - \frac{\sigma \bar{y}}{w} = \frac{\omega_0}{\omega_1 h N}, \tag{A21}$$

Substituting $\bar{y} = (1 + r)\bar{k} + hwT_0$ into (A21), we get

$$\tilde{w}_1 \bar{k} + \tilde{w}_0 = \left(\theta - \frac{\bar{k}}{\omega_1 h N}\right) \frac{\upsilon (1 + r)\bar{k} + \upsilon h w T_0}{\tilde{w}}, \tag{A22}$$

where

$$\tilde{w}_1(z, X, N) \equiv \frac{\bar{r}}{\omega_1 h N} - \frac{(1 + r)\sigma}{w}, \quad \tilde{w}_0(z, X, N) \equiv (1 - \sigma h)T_0 - \frac{\omega_0}{\omega_1 h N}.$$

Insert $\tilde{w} = \bar{k} + h\theta w$ in (A22)

$$\bar{k}^2 + \tilde{\omega}_1 \bar{k} - \tilde{\omega}_0 = 0, \tag{A23}$$

where

$$\tilde{w}_2 \equiv \tilde{w}_1 + \frac{\upsilon (1 + r)}{\omega_1 h N}, \quad \tilde{\omega}_1(z, X, N) \equiv$$
$$\frac{\tilde{w}_0 + h\theta w \tilde{w}_1 - \theta \upsilon (1 + r) + \upsilon w T_0 / \omega_1 N}{\tilde{w}_2}, \quad \tilde{\omega}_0(z, X, N) \equiv \frac{(\upsilon T_0 - \tilde{w}_0) h \theta w}{\tilde{w}_2}.$$

Solve (A23) with regard to $\bar{k}$

$$\bar{k}_{1,2} = \phi(z, X, N) \equiv -\frac{\tilde{\omega}_1}{2} \pm \sqrt{\frac{\tilde{\omega}_1^2}{4} + \tilde{\omega}_0}. \tag{A24}$$

We determine all the variables as functions of $z(t)$, $X(t)$, and $N(t)$ at any point of time by the following procedure: $\bar{k}$ by (A24) $\to N_i$ and N_x by (A15) $\to$ r, w and p by (A6) $\to$ K_i and K_x by (A1) $\to$ $\bar{y} = (1 + r)\bar{k} + hwT_0 \to c$, s, $\tilde{T}$, c_x, and n by (14) $\to K_R$ by (A3) $\to \bar{T}$ by (A10) $\to T$ by (A12) $\to$ $\bar{N}$ by (1) $\to$ F_i by (3) $\to$ F_x by (6).

From this procedure, (5) and (18), it is straightforward to show that the motion of resource stock and the population can be expressed as function of $z(t)$, $X(t)$, and $N(t)$ at any point in time

$$\dot{X} = \Omega_X(z, X, N),$$
$$\dot{N} = \Omega_N(z, X, N). \tag{A25}$$

We now show that change in $z(t)$ can also be expressed as a differential equation in terms of $z(t)$, $X(t)$, and $N(t)$. From (19), we have

$$\dot{\bar{k}} = \Omega_0(z, X, N) \equiv \lambda \bar{y} - \bar{k}. \tag{A26}$$

Taking derivatives of (A24) for one of the two solutions with respect to time, we have

$$\dot{k} = \frac{\partial \phi}{\partial z}\dot{z} + \frac{\partial \phi}{\partial X}\Omega_X + \frac{\partial \phi}{\partial X}\Omega_N, \tag{A27}$$

where we also use (A25). From (A26) and (A27), we solve

$$\dot{z} = \Omega_z(z, X, N) \equiv \left[\Omega_0 - \frac{\partial \phi}{\partial X}\Omega_X - \frac{\partial \phi}{\partial N}\Omega_N\right]\left(\frac{\partial \phi}{\partial z}\right)^{-1}. \tag{A28}$$

The three differential equations (A25) and (A28) contain three variables $z(t)$, $X(t)$, and $N(t)$. We thus proved the lemma.

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